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MATHEMATICS
TARGET IIT JEE

LIMIT

THEORY AND EXERCISE BOOKLET

CONTENTS

JEE Syllabus :

limit and continuity of a function, limit and continuity of the sum, difference, product and quotient of two functions, even and odd functions, inverse of a function.

A. DEFINITION OF LIMIT

Let f be a function defined on an open interval containing c (except possibly at c) and let L be a real number. The statement $\lim\limits_{x \to c} f(x) = L$ means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta, \text{ then } |f(x) - L| < \varepsilon.$$

B. THE EXISTENCE OF A LIMIT

Let f be a function and let c and L be real numbers. The limit of f(x) as x approaches c is L if and only if

$$\lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

In other words limit of a function f(x) is said to exist as, $x \to a$ when

$$\underset{x \to a^-}{\text{Limit}} f(x) = \underset{x \to a^+}{\text{Limit}} f(x) = \text{A finite quantity .}$$

$$f(a^-) = f(a^+)$$

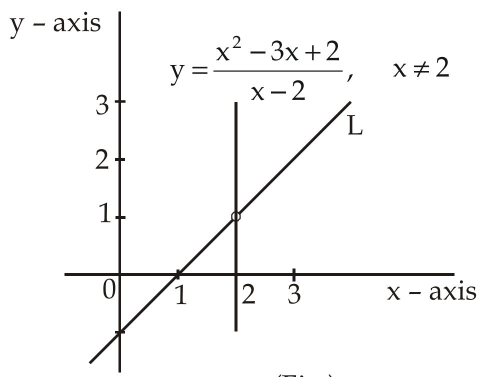
$$\begin{pmatrix}\text{Left hand limit}\\ \text{L.H.L.}\end{pmatrix} \qquad \begin{pmatrix}\text{Right hand limit}\\ \text{R.H.L.}\end{pmatrix}$$

Ex.1 Consider the function f defined by $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$, $x \neq 2$

Sol. The domain of f is the set of all real numbers with the exception of the number 2, which has been excluded because substitution of x = 2 in the expression for f(x) yields the undefined term $\dfrac{0}{0}$. On the other hand, $x^2 - 3x + 2 = (x - 1)(x - 2)$ and

$$f(x) = \frac{(x-1)(x-2)}{x-2} = x - 1, \qquad \text{provided } x \neq 2 \quad ......(1)$$

The graph of the function y = x – 1 a straight line L; so the graph of f(x) is the punctured line from L by omitting the one point (2, 1) (See Fig.)



(Fig.)

Although the function f is not defined at x = 2, we know its behaviour for values of x near 2. The graph makes it clear that if x is close to 2, then f(x) is close to 1. In fact, the values of f(x) can be brought arbitrarily close to 1 by taking x sufficiently close to 2. We express this fact by writing.

$$\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2} = 1,$$

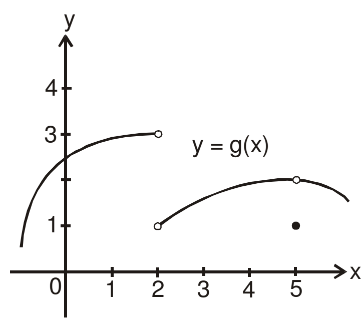
which means that the limit of $\dfrac{x^2 - 3x + 2}{x - 2}$ is 1 as x approaches 2.

Ex.2 The graph of a function g is shown in the figure. Use it to state the values (if they exist) of the following

(a) $\underset{x \to 2^-}{\text{Lim}} g(x)$ **(b)** $\underset{x \to 2^+}{\text{Lim}} g(x)$

(c) $\underset{x \to 2}{\text{Lim}} g(x)$ **(d)** $\underset{x \to 5^-}{\text{Lim}} g(x)$

(e) $\underset{x \to 5^+}{\text{Lim}} g(x)$ **(f)** $\underset{x \to 5}{\text{Lim}} g(x)$



Sol. From the graph we see that the values of g(x) approach 3 as x approaches 2 from the left, but they approach 1 as x approaches 2 from the right. Therefore

(a) $\lim\limits_{x\to 2^-} g(x) = 3$ and **(b)** $\lim\limits_{x\to 2^+} g(x) = 1$

(c) Since the left and right limits are different, we conclude that $\lim\limits_{x\to 2} g(x)$ does not exist.

The graph also show that

(d) $\lim\limits_{x\to 5^-} g(x) = 2$ and **(e)** $\lim\limits_{x\to 5^+} g(x) = 2$

(f) This time the left and right limits are the same and so, we have $\lim\limits_{x\to 5} g(x) = 2$

Despite this fact, notice that $g(5) \neq 2$.

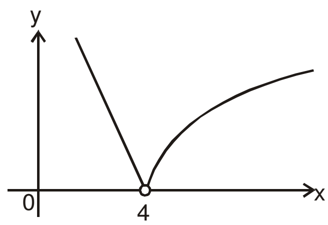
Ex.3 If $f(x) = \begin{cases} \sqrt{x-4} & \text{if} \quad x > 4 \\ 8-2x & \text{if} \quad x < 4 \end{cases}$ determine whether $\lim\limits_{x\to 4} f(x)$ exists.

Sol. Since f(x) = for x > 4, we have

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \sqrt{x-4} = \sqrt{4-4} = 0$$

Since f(x) = 8 – 2x for x < 4, we have

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (8 - 2x) = 8 - 2.4 = 0$$



The right and left-hand limits are equal. Thus, the limit exists and $\lim\limits_{x\to 4} f(x) = 0$

The graph of f is shown in the figure.

Ex.4 Evaluate the left hand and right hand limits of the function $f(x) = \begin{cases} \dfrac{\sqrt{(x^2-6x+9)}}{(x-3)} & , x \neq 3 \\ 0 & , x = 3 \end{cases}$ at x = 3.

Sol. The given function can be written as $\begin{cases} \dfrac{|x-3|}{(x-3)} & , x \neq 3 \\ 0 & , x = 3 \end{cases}$

$$\therefore \quad \text{L.H.L.} = \lim_{x\to 3-} f(x) = \lim_{h\to 0} (3 - h) = \lim_{h\to 0} \frac{|3-h-3|}{(3-h-3)} = \lim_{h\to 0} \frac{|-h|}{(-h)} = \lim_{h\to 0} \frac{h}{-h} = -1$$

$$\text{and R.H.L.} = \lim_{x\to 3+} f(x) = \lim_{h\to 0} (3 + h) = \lim_{h\to 0} \frac{|3+h-3|}{(3+h-3)} = \lim_{h\to 0} \frac{|h|}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

Hence left hand limit and right hand limit of f(x) at x = 3 are –1 and 1 respectively.

Ex.5 Let $f(x) = \begin{cases} \cos[x], & x \geq 0 \\ |x| + a, & x < 0 \end{cases}$. Find the value of a, given that $\lim\limits_{x\to 0} f(x)$ exists.

(where [*] denotes the greatest integer function)

Sol. Since $\lim\limits_{x\to 0} f(x)$ exists $\Rightarrow \lim\limits_{x\to 0-} f(x) = \lim\limits_{x\to 0+} f(x) \Rightarrow \lim\limits_{h\to 0} f(0 - h) = \lim\limits_{h\to 0} f(0 + h)$

$$\Rightarrow \lim_{h\to 0} |0 - h| + a = \lim_{h\to 0} \cos[0 + h] \quad \Rightarrow \quad a = \cos 0 = 1 \quad \therefore \quad a = 1$$

Ex.6 Evaluate $\lim\limits_{x\to 0}\left[\dfrac{\sin^{-1}x}{x}\right]$ (where [*] denotes the greatest integer function)

Sol. Let P = $\lim\limits_{x\to 0}\left[\dfrac{\sin^{-1}x}{x}\right]$ for $x > 0$, $\sin^{-1}x > x \Rightarrow \dfrac{\sin^{-1}x}{x} > 1$

$\therefore$ R.H.L. $= \lim\limits_{x\to 0}\left[\dfrac{\sin^{-1}x}{x}\right] = 1$ for $x < 0$, $\sin^{-1}x > x \Rightarrow \dfrac{\sin^{-1}x}{x} > 1$

$\therefore$ L.H.L. $= \lim\limits_{x\to 0}\left[\dfrac{\sin^{-1}x}{x}\right] = 1$ Hence P = 1.

Ex.7 Solve $\lim\limits_{x\to 0}\left[\sin\dfrac{|x|}{x}\right]$, (where [*] denotes the greatest integer function)

Sol. Here, $\lim\limits_{x\to 0}\left[\sin\dfrac{|x|}{x}\right]$, since we have greatest integer function we must define function.

Now, RHL (put x = 0 + h) $\lim\limits_{h\to 0}\left[\dfrac{\sin|0+h|}{0+h}\right]$, we know $\dfrac{\sin h}{h} \to 1$ as $h \to 0$ but less than 1.

$\therefore \lim\limits_{h\to 0} 0 = 0$ $\{\because \left[\dfrac{\sin h}{h}\right] = 0$ as $h \to 0\}$ $\Rightarrow$ RHL = 0

again, LHL (put x = 0 – h) $\lim\limits_{h\to 0}\left[\sin\dfrac{|0-h|}{0-h}\right]$, we know $\dfrac{\sin h}{-h} \to -1$ as $h \to 0$ but greater than -1.

$\therefore \lim\limits_{h\to 0} -1 = -1$ $\{\because \left[-\dfrac{\sin h}{h}\right] = -1$ as $h \to 0\}$ $\Rightarrow$ LHL = -1

$\therefore$ limit does not exists as RHL = 0 and LHL = -1

C. FUNDAMENTAL THEOREMS ON LIMITS

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits. $\lim\limits_{x\to c} f(x) = L$ and $\lim\limits_{x\to c} g(x) = K$

1. **Sum Rule :** $\lim\limits_{x\to c}[f(x) + g(x)] = L + K$
2. **Difference Rule :** $\lim\limits_{x\to c}[f(x) - g(x)] = L - K$
3. **Product Rule :** $\lim\limits_{x\to c}[f(x)\, g(x)] = LK$
4. **Quotient Rule :** $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{L}{K}$, provided $K \neq 0$
5. **Constant Multiplication Rule :** $\lim\limits_{x\to c}[b\, f(x)] = bL$
6. **Composition Rule :** $\underset{x\to a}{\text{Limit}}\; f(g(x)) = f\left(\underset{x\to a}{\text{Limit}}\, g(x)\right) = f(m)$; provided f is continuous at x = m.

For example $\underset{x\to a}{\text{Limit}}\; \ln(f(x)) = \ln\left(\underset{x\to a}{\text{Limit}}\, f(x)\right) = \ln l\,(l > 0)$.

Ex.8 Evaluate the following limits and justify each step.

(a) $\lim\limits_{x\to5}(2x^2 - 3x + 4)$ **(b)** $\lim\limits_{x\to-2}\dfrac{x^3+2x^2-1}{5-3x}$

Sol. **(a)** $\lim\limits_{x\to5}(2x^2 - 3x + 4) = \lim\limits_{x\to5}(2x^2) - \lim\limits_{x\to5}(3x) + \lim\limits_{x\to5} 4$

$$= 2\lim_{x\to5} x^2 - 3\lim_{x\to5} x + \lim_{x\to5} 4 = 2(5^2) - 3(5) + 4 = 39$$

(b) We start by using laws of limit, but their use is fully justified only at the final stage when we see that the limits of the numerator and denominator exist and the limit of the denominator is not 0.

$$\lim_{x\to-2}\frac{x^3+2x^2-1}{5-3x} = \frac{\lim\limits_{x\to-2}(x^3+2x^2-1)}{\lim\limits_{x\to-2}(5-3x)} = \frac{\lim\limits_{x\to-2}x^3+\lim\limits_{x\to-2}2x^2-\lim\limits_{x\to-2}1}{\lim\limits_{x\to-2}5-3\lim\limits_{x\to-2}x} = \frac{(-2)^3+2(-2)^2-1}{5-3(-2)} = -\frac{1}{11}.$$

D. NON-EXISTENCE OF LIMIT

Three of the most common types of behaviour associated with the non-existence of a limit.

1. f(x) approaches a different number from the right side of c than it approaches from the left side.

2. f(x) increases or decreases without bound as x approaches c.

3. f(x) oscillates between two fixed values as x approaches c.

There are many other interesting functions that have unusual limit behaviour. An often cited one is the Dirichlet function $f(x) = \begin{cases}0, & \text{if } x \text{ is rational}\\ 1, & \text{if } x \text{ is irrational.}\end{cases}$. This function has no limit at any real number c.

E. INDETERMINANT FORMS : $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^0, \infty^0$ and 1^∞

Ex.9 Which of the following limits are in indeterminant forms. Also indicate the form

(i) $\lim\limits_{x\to0}\dfrac{1}{x}$ **(ii)** $\lim\limits_{x\to0}\dfrac{1-x}{1-x^2}$ **(iii)** $\lim\limits_{x\to0} x\,\ell n\, x$ **(iv)** $\lim\limits_{x\to0}\left(\dfrac{1}{x}-\dfrac{1}{x^2}\right)$

(v) $\lim\limits_{x\to0}(\sin x)^x$ **(vi)** $\lim\limits_{x\to0}(\ell n\, x)^x$ **(vii)** $\lim\limits_{x\to0}(1+\sin x)^{\frac{1}{x}}$ **(viii)** $\lim\limits_{x\to0}(1)^{\frac{1}{x}}$

Sol. **(i)** No **(ii)** Yes $\frac{0}{0}$ form **(iii)** Yes $0\times\infty$ form **(iv)** Yes $(\infty - \infty)$ form

(v) Yes, 0^0 form **(vi)** Yes ∞^0 form **(vii)** Yes 1^∞ form **(viii)** No

Remark :

(i) '0' doesn't means exact zero but represent a value approaching towards zero similarly to '1' and infinity.

(ii) $\infty + \infty = \infty$ **(iii)** $\infty\times\infty = \infty$ **(iv)** $(a/\infty) = 0$ if a is finite

(v) $\dfrac{a}{0}$ is not defined for any $a \in R$. **(vi)** a b = 0, if & only if a = 0 or b = 0 and a & b are finite.

F. METHODS OF EVALUATING LIMITS

(Rationalization, Factorization and Cancellation of Common Factors)

Ex.10 Evaluate $\lim\limits_{x\to 1} \arcsin\left(\dfrac{1-\sqrt{x}}{1-x}\right)$

Sol. $\lim\limits_{x\to 1} \sin^{-1}\left(\dfrac{1-\sqrt{x}}{1-x}\right) = \sin^{-1}\left(\lim\limits_{x\to 1}\dfrac{1-\sqrt{x}}{1-x}\right) = \sin^{-1}\left(\lim\limits_{x\to 1}\dfrac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})}\right) = \sin^{-1}\left(\lim\limits_{x\to 1}\dfrac{1}{1+\sqrt{x}}\right) = \sin^{-1}\dfrac{1}{2} = \dfrac{\pi}{6}$

Ex.11 Evaluate $\lim\limits_{x\to 2}\dfrac{x-2}{\sqrt{x^2-4}-\sqrt{x-2}}$

Sol. $L = \lim\limits_{x\to 2}\dfrac{x-2}{\sqrt{x^2-4}+\sqrt{x-2}}\cdot\dfrac{(\sqrt{x^2-4}-\sqrt{x-2})}{(\sqrt{x^2-4}-\sqrt{x-2})} = \lim\limits_{x\to 2}\dfrac{(x-2)(\sqrt{x^2-4}-\sqrt{x-2})}{(x^2-4)-(x-2)}$

$= \lim\limits_{x\to 2}\dfrac{(x-2)(\sqrt{x^2-4}-\sqrt{x-2})}{x^2-x-2} = \lim\limits_{x\to 2}\dfrac{(x-2)(\sqrt{x^2-4}-\sqrt{x-2})}{(x-2)(x+1)} = \lim\limits_{x\to 2}\dfrac{\sqrt{x^2-4}\,\sqrt{x-2}}{x+1} = 0.$

Ex.12 Evaluate : $\lim\limits_{x\to -1}\dfrac{\sqrt[3]{(7-x)}-2}{(x+1)}$

Sol. $\sqrt[3]{(7-x)} - 2 = \dfrac{(7-x)-8}{(7-x)^{2/3}+(7-x)^{1/3}.2+4} = \dfrac{(x+1)}{(x+1)^{2/3}+(x+1)^{1/3}.2+4}$ ...(1)

$\therefore \lim\limits_{x\to -1}\dfrac{\sqrt[3]{(7-x)}-2}{(x+1)} = -\lim\limits_{x\to -1}\dfrac{(x+1)}{(x+1)(7-x)^{2/3}+(7-x)^{1/3}.2+4}$ [From (1)]

$= \lim\limits_{x\to -1}\dfrac{1}{(7-x)^{2/3}+(7-x)^{1/3}.2+4} = \dfrac{1}{4+2+4} = -\dfrac{1}{10}.$

Ex.13 Evaluate : $\lim\limits_{x\to 0}\dfrac{\sqrt{(1+\sin 3x)}-1}{\ln(1+\tan 2x)}$

Sol. $\lim\limits_{x\to 0}\dfrac{\sqrt{(1+\sin 3x)}-1}{\ln(1+\tan 2x)}$ $\left(\text{form}\dfrac{0}{0}\right) = \lim\limits_{x\to 0}\dfrac{(\sqrt{(1+\sin 3x)}-1)(\sqrt{(1+\sin 3x)}+1)}{\ln(1+\tan 2x)(\sqrt{(1+\sin 3x)}+1)}$

$= \lim\limits_{x\to 0}\dfrac{\sin 3x}{3x}\ .\ \lim\limits_{x\to 0}\dfrac{\tan 2x}{\ln(1+\tan 2x)}\ .\ \lim\limits_{x\to 0}\dfrac{3x}{\tan 2x}\ .\ \lim\limits_{x\to 0}\dfrac{\tan 2x}{(\sqrt{1+\sin 3x})+1)} = 1\ .\ 1\ .\ \dfrac{3}{2}\ .\ \dfrac{1}{(1+1)} = \dfrac{3}{4}.$

Ex.14 Evaluate $\underset{x\to a^-}{\text{Limit}}\left[\sqrt{a^2-x^2}.\cot\left\{\dfrac{\pi}{2}.\sqrt{\dfrac{a-x}{a+x}}\right\}\right]$.

Sol. $\underset{x\to a^-}{\text{Limit}}\left[\sqrt{a^2-x^2}.\cot\left\{\dfrac{\pi}{2}.\sqrt{\dfrac{a-x}{a+x}}\right\}\right] = \underset{h\to 0}{\text{Limit}}\ \sqrt{a^2-(a-h)^2}\ \cot\left\{\dfrac{\pi}{2}\sqrt{\dfrac{h}{2a-h}}\right\}$

$= \underset{h\to 0}{\text{Limit}}\ \dfrac{\sqrt{2ah-h^2}}{\tan\dfrac{\pi}{2}\sqrt{\dfrac{h}{2a-h}}} = \underset{h\to 0}{\text{Limit}}\ \dfrac{\sqrt{h}\sqrt{2a-h}}{\tan\dfrac{\pi}{2}\sqrt{\dfrac{h}{2a-h}}}\ \dfrac{\dfrac{\pi}{2}\sqrt{\dfrac{h}{2a-h}}}{\dfrac{\pi}{2}\sqrt{\dfrac{h}{2a-h}}}$

$= \underset{h\to 0}{\text{Limit}}\ \sqrt{h}\sqrt{2a-h}\times\dfrac{2\sqrt{2a-h}}{\pi\sqrt{h}} = \dfrac{4a}{\pi}$

Ex.15 Find $\lim_{n\to\infty}\left(\frac{2n^3}{2n^2+3}+\frac{1+5n^2}{5n+1}\right)$

Sol. $\lim_{n\to\infty}\left(\frac{2n^3-13n^2+3}{10n^3+2n^2+15n+3}\right)$ $\lim_{n\to\infty}\left(\frac{2-13/n+3/n^3}{10n+2n+15/n^2+3/n^3}\right) = \frac{1}{5}$

Ex.16 Find $\lim_{n\to\infty}\frac{\sqrt{n^2+1}+\sqrt{n}}{4\sqrt{n^3+1}-\sqrt{n}}$

Sol. $\lim_{n\to\infty}\frac{(n)\left[\sqrt{1+1/n^2}+\sqrt{1/n}\right]}{n^{3/4}\left(\sqrt{1+1/n^2+1}-\sqrt{1/n}\right)} = \lim_{n\to\infty} n^{1/4}\frac{\left(\sqrt{1+1/n^2}+\sqrt{1/n}\right)}{\left(\sqrt{1+1/n^2}-\sqrt{1/n}\right)} = \infty$

Ex.17 Find $\lim_{n\to\infty}\frac{2n}{2n^2-1}\cos\frac{n+1}{2n-1}-\frac{n}{1-2n}\times\frac{n(-1)^n}{n^2+1}$

Sol. $\lim_{\substack{n\to\infty\\1/n\to 0}}\left(\frac{2}{2-1/n^2}\right)\frac{1}{2}\cos\left(\frac{1+1/n}{2n-1/n}\right)-\frac{1}{(1/n-2)}\times\frac{(-1)^n}{(1+1/n^2)}\times\frac{1}{n}$

$= \lim_{\substack{n\to\infty\\1/n\to 0}}\frac{1}{n}\left[\frac{2}{2-1/n^2}\times\cos\left(\frac{1+1/n}{2-1/n}\right)-\frac{1}{(1/n-2)}\times\frac{(-1)^n}{(1+1/n^2)}\right] = 0\times\left[\frac{2}{2}\times\cos\frac{1}{2}+\frac{1}{2}\times\frac{1}{1}\right] = 0.$

Ex.18 Find $\lim_{x\to\infty}\left(\frac{1-2x}{(1+8x^3)^{1/3}}+2^{-x^7}\right)$

Sol. $\lim_{\substack{x\to\infty\\1/x\to 0}}\frac{x\left(\frac{1}{2}-2\right)}{\left(\frac{1}{x^3}+8\right)^{1/3}}+\frac{1}{2^{x^7}} = \frac{-2}{(8)^{1/3}} = -1$

Ex.19 Evaluate a, b, c and d, if $\lim_{x\to\infty}(\sqrt{x^4+ax^3+3x^2+bx+2}-\sqrt{x^4+2x^3-cx^2+3x-d}) = 4$.

Sol. Here, $\lim_{x\to\infty}(\sqrt{x^4+ax^3+3x^2+bx+2}-\sqrt{x^4+2x^3-cx^2+3x-d}) = 4$ $(\infty-\infty \text{ form})$

Rationalizing $\Rightarrow \lim_{x\to\infty}\frac{(a-2)x^3+(3+c)x^2+(b-3)x+(2+d)}{\sqrt{x^4+ax^3+3x^2+bx+2}+\sqrt{x^4+2x^3-cx^2+3x-d}}$

Since, limit is finite, so the degree of the numerator must be 2. So, a – 2 = 0 i.e. a = 2.

$\Rightarrow \lim_{x\to\infty}\frac{(3+c)x^2+(b-3)x+(2+d)}{\sqrt{x^4+ax^3+3x^2+bx+2}+\sqrt{x^4+2x^3-cx^2+3x-d'}}$

dividing numerator and denominator by x^2 .We get,

$$\Rightarrow \lim_{x\to\infty} \frac{(3+c)+(b-3)/x+(2+d)/x^2}{\sqrt{1+\frac{a}{x}+\frac{3}{x^2}+\frac{b}{x^3}+\frac{2}{x^4}}+\sqrt{1+\frac{2}{x}-\frac{c}{x^2}+\frac{3}{x^3}-\frac{d}{x^4}}}$$

$\Rightarrow \frac{3+c}{2}$ given, $\frac{3+c}{2} = 4 \Rightarrow$ c = 5 $\therefore$ c = 5, a = 2

Hence, a = 2, c = 5 and b, d are real numbers.

Ex.20 Evaluate $\lim_{n\to\infty} n^2\left\{\sqrt{\left(1-\cos\frac{1}{n}\right)\sqrt{\left(1-\cos\frac{1}{n}\right)\sqrt{\left(1-\cos\frac{1}{n}\right)\ldots\infty}}}\right\}$

Sol. P = $\lim_{n\to\infty} n^2\left\{\sqrt{\left(1-\cos\frac{1}{n}\right)\sqrt{\left(1-\cos\frac{1}{n}\right)\sqrt{\left(1-\cos\frac{1}{n}\right)\ldots\infty}}}\right\}$ Put $\frac{1}{n}$ = x

$\therefore$ P = $\lim_{x\to 0} \frac{\sqrt{(1-\cos x)\sqrt{(1-\cos x)\sqrt{(1-\cos x)\ldots..\infty}}}}{x^2}$ = $\lim_{x\to 0} \frac{(1-\cos x)^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}-\ldots..\infty}}{x^2}$

= $\lim_{x\to 0} \frac{(1-\cos x)}{x^2}\cdot\frac{(1+\cos x)}{(1+\cos x)}$ = $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 \cdot \lim_{x\to 0}\frac{1}{(1+\cos x)}$ = $(1)^2 \cdot \frac{1}{1+1} = \frac{1}{2}$

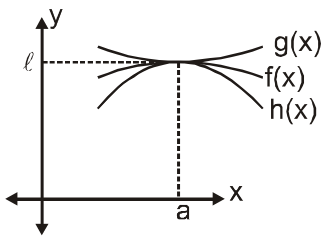
G. STANDARD THEOREM

(1) SANDWICH THEOREM / SQUEEZE PLAY THEOREM

Statement : If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$\lim_{x\to c} h(x) = \ell = \lim_{x\to c} g(x)$

then $\lim_{x\to c} f(x)$ exists and is equal to ℓ.



Ex.21 Find $\lim_{x\to 0} f(x)$ if $f(x) = \begin{cases} 1+2x^2 & , \ x \in Q \\ 1+x^4 & , \ x \notin Q \end{cases}$

Sol. Consider $|x| \le 1$. $x^4 \le x^2 \le 2x^2$.

Now $1 \le f(x) \le 1 + 2x^2 \ \forall$ x.

By Sandwich theorem, $\lim_{x\to 1} 1 + 2x^2 = 1$. Hence $\lim_{x\to 0} f(x) = 1$.

Ex.22 Let a function f(x) be such that $|f(x)| \le M$ for any $x \ne 0$. Prove that $\lim_{x\to 0} x f(x) = 0$

Sol. We have $|x f(x)| \le |x| . M \Rightarrow -M|x| \le x f(x) \le M|x|$ for any $x \ne 0$.

Since $\lim_{x\to 0} M|x| = 0$ and $\lim_{x\to 0} -M|x| = 0$, by Sandwich theorem, $\lim_{x\to 0} x f(x) = 0$.

Ex.23 Use Sandwich theorem to evaluate: $\lim\limits_{n\to\infty} \sum\limits_{k=n^2}^{(n+1)^2} \frac{1}{\sqrt{k}}$

Sol. $f(n) = \underbrace{\frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{(n^2+2)}} + \ldots\ldots\ldots + \frac{1}{\sqrt{n^2+2n+1}}}_{(2n+2)\text{ terms}}$

terms of the sequence are decreasing and number of terms are (2n + 2)

$\frac{2n+2}{\sqrt{n^2+2n+1}} \le f(n) \le \frac{2n+2}{\sqrt{n^2}}$ now $\lim\limits_{n\to\infty} \frac{2(n+1)}{\sqrt{n^2+2n+1}} = \lim\limits_{n\to\infty} \frac{2n\left(1+\frac{1}{n}\right)}{n\sqrt{1+\frac{2}{n}+\frac{1}{n^2}}}$;

|||ly $\lim\limits_{n\to\infty} \frac{2(n+1)}{\sqrt{n^2}} = \lim\limits_{n\to\infty} \frac{2(n+1)}{n} = 2$ $\therefore$ $\lim\limits_{n\to\infty} f(x) = 2$

Ex.24 Suppose that function f : R → R satisfies the inequality, $\left|\sum\limits_{k=1}^{n} 3^k \{f(x+ky) - f(x-ky)\}\right| \le 1$

for every positive integer n and for all x, y $\in$ R. Prove that f is a constant function.

Sol. Replacing n by (n – 1) in, $\left|\sum\limits_{k=1}^{n} 3^k \{f(x+ky) - f(x-ky)\}\right| \le 1$ ...(i)

we get; $\left|\sum\limits_{k=1}^{n-1} 3^k \{f(x+ky) - f(x-ky)\}\right| \le 1$ ...(ii)

Subtracting (i) and (ii) we get, $|3^n \{f(x+ny) - f(x-ny)\}| \le 2$

$\Rightarrow$ $|\{f(x+ny) - f(x-ny)\}| \le \frac{2}{3^n}$ ...(iii)

We choose x and y such that x + ny = u and x – ny = v, where u, v $\in$ R and n $\in$ N.

(iii) becomes, $|f(u) - f(v)| \le \frac{2}{3^n}$ for arbitrary n $\in$ N

i.e., as $n \to \infty$ $\Rightarrow$ $|f(u) - f(v)| \le \lim\limits_{n\to\infty} \frac{2}{3^n}$ $\Rightarrow$ $|f(u) - f(v)| \le 0$ $\Rightarrow$ f(u) = f(v) Hence f is constant.

(2) LIMIT OF TRIGONOMETRIC FUNCTIONS

(a) $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$ (approach from left side)

(b) $\lim\limits_{x\to 0} \frac{\tan x}{x} = 1$ (approach from right side)

(c) $\lim\limits_{x\to 0} \frac{\tan^{-1} x}{x} = 1$

(d) $\lim\limits_{x\to 0} \frac{\sin^{-1} x}{x} = 1$

[Where x is measured in radians]

Ex.25 Find $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}$

Sol. $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}=\lim\limits_{x\to 0}\dfrac{\dfrac{\sin x}{\cos x}-\sin x}{x^3}=\lim\limits_{x\to 0}\dfrac{(1-\cos x)}{x^3\cos x}=\lim\limits_{x\to 0}\dfrac{\sin x.2\sin^2\dfrac{x}{2}}{x^3\cos x}$

$$=\lim_{x\to 0}\frac{2\left(\dfrac{\sin x}{x}\right)x.\left(\dfrac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\left(\dfrac{x}{2}\right)^2}{x^3\cos x}=\lim_{x\to 0}\frac{2\left(\dfrac{\sin x}{x}\right).\left(\dfrac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2.\dfrac{1}{4}}{\cos x}\ \frac{2.1.1^2.\frac{1}{4}}{1}=\frac{1}{2}$$

Ex.26 Evaluate : $\lim\limits_{\theta\to\pi/4}\dfrac{(1-\tan\theta)}{(1-\sqrt{2}\sin\theta)}$

Sol. Let P = $\lim\limits_{x\to 0}\dfrac{(1-\tan\theta)}{(1-\sqrt{2}\sin\theta)}$ $\left(\text{form}\dfrac{0}{0}\right)$ Put $\theta=\dfrac{\pi}{4}$

$$\therefore\quad P=\lim_{h\to\pi/4}\frac{1-\tan\left(\frac{\pi}{4}+h\right)}{1-\sqrt{2}\sin\left(\frac{\pi}{4}+h\right)}=\lim_{h\to 0}\frac{1-\left(\dfrac{1+\tan h}{1-\tan h}\right)}{1-\sqrt{2}\left(\dfrac{1}{\sqrt{2}}\cos h+\dfrac{1}{\sqrt{2}}\sin h\right)}\ \left(\text{form}\frac{0}{0}\right)$$

$$=\lim_{h\to 0}\frac{-2\tan h}{(1-\tan h)(1-\cos h-\sin h)}=\lim_{h\to 0}\frac{-2\tan h}{(1-\tan h)\left(\dfrac{(1-\cos h)(1+\cos h)}{(1+\cos h)}-\sin h\right)}$$

$$=-2\lim_{h\to 0}\frac{\tan h}{(1-\tan h)\left(\dfrac{\sin^2 h}{1+\cos h}-\sin h\right)}=-2\lim_{h\to 0}\frac{\sin h}{(1-\cos h-\sin h)\left(\dfrac{\sin^2 h}{1+\cos h}-\sin h\right)}$$

$$=-2\lim_{h\to 0}\frac{1}{(\cos h-\sin h)\left(\dfrac{\sin h}{1+\cos h}-1\right)}=-2.\frac{1}{(1-0)(0-1)}=2.$$

Ex. 27 Solve $\lim\limits_{x\to\frac{\pi}{2}}\sqrt{\dfrac{\tan x-\sin\{\tan^{-1}(\tan x)\}}{\tan x+\cos^2(\tan x)}}$

Sol. Here, RHL at $x=\dfrac{\pi}{2}$

$$=\lim_{x\to\frac{\pi}{2}^+}\sqrt{\frac{\tan x-\sin\{\tan^{-1}(\tan x)\}}{\tan x+\cos^2(\tan x)}}=\lim_{x\to\frac{\pi}{2}^+}\sqrt{\frac{\tan x-\sin(x-\pi)}{\tan x+\cos^2(\tan x)}}\quad\left\{\because \tan^{-1}(\tan x)=x-\pi,\text{ when } x>\frac{\pi}{2}\right\}$$

$$= \lim_{x \to \frac{\pi}{2}^+} \sqrt{\frac{1 + \frac{\sin x}{\tan x}}{1 + \frac{\cos^2(\tan x)}{\tan x}}} \qquad = \sqrt{\frac{1+0}{1+0}} = 1$$

again, LHL at $x = \frac{\pi}{2}$ $\qquad = \lim_{x \to \frac{\pi}{2}^-} \sqrt{\frac{\tan x - \sin\{\tan^{-1}(\tan x)\}}{\tan x + \cos^2(\tan x)}}$

$$= \lim_{x \to \frac{\pi}{2}^-} \sqrt{\frac{\tan x - \sin(x)}{\tan x + \cos^2(\tan x)}} \qquad \left\{\text{as } \tan^{-1}(\tan x) = x, \text{ when } x < \frac{\pi}{2}\right\}$$

$$= \lim_{x \to \frac{\pi}{2}^-} \sqrt{\frac{1 - \frac{\sin x}{\tan x}}{1 + \frac{\cos^2(\tan x)}{\tan x}}} = \sqrt{\frac{1+0}{1+0}} = 1 \quad \therefore \lim_{x \to \frac{\pi}{2}} \sqrt{\frac{\tan x - \sin\{\tan^{-1}(\tan x)\}}{\tan x + \cos^2(\tan x)}} = 1$$

Ex.28 Let $f(x) = \frac{\sin^{-1}(1-\{x\}).\cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}.(1-\{x\})}$, then find $\lim_{x \to 0+} f(x)$ and $\lim_{x \to 0-} f(x)$.

(where $\{x\}$ denotes the fractional part of x)

Sol. We have $f(x) = \frac{\sin^{-1}(1-\{x\}).\cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}.(1-\{x\})}$

$$\therefore \quad \lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} \frac{\sin^{-1}(1-\{0+h\}).\cos^{-1}(1-\{0+h\})}{\sqrt{2\{0+h\}}.(1-\{0+h\})} = \lim_{h \to 0} \frac{\sin^{-1}(1-h).\cos^{-1}(1-h)}{\sqrt{2h}.(1-h)}$$

$$= \lim_{h \to 0} \frac{\sin^{-1}(1-h)}{(1-h)} \cdot \lim_{h \to 0} \frac{\cos^{-1}(1-h)}{\sqrt{2h}}$$

in second limit put $1 - h = \cos\theta$

$$= \lim_{h \to 0} \frac{\sin^{-1}(1-h)}{(1-h)} \cdot \lim_{h \to 0} \frac{\cos^{-1}(\cos\theta)}{\sqrt{2(1-\cos\theta)}} = \lim_{h \to 0} \frac{\sin^{-1}(1-h)}{(1-h)} \cdot \lim_{h \to 0} \frac{\theta}{2\sin(\theta/2)} \qquad (\because \theta > 0)$$

$= \sin^{-1} 1 \,.\, 1 = \pi/2$

and $\lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} \frac{\sin^{-1}(1-\{0-h\}).\cos^{-1}(1-\{0-h\})}{\sqrt{2\{0-h\}}.(1-\{0-h\})}$

$$= \lim_{h \to 0} \frac{\sin^{-1}(1+h-1).\cos^{-1}(1+h+1)}{\sqrt{2(-h+1)}.(1+h-1)} = \lim_{h \to 0} \frac{\sin^{-1} h}{h} \cdot \lim_{h \to 0} \frac{\cos^{-1} h}{\sqrt{2(1-h)}} \qquad = 1 \cdot \frac{\pi/2}{\sqrt{2}} = \frac{\pi}{2\sqrt{2}}$$

NOTE : LIMIT USING EXPANSION OF FUNCTIONS

(i) $a^x = 1 + \frac{x\,\ell na}{1!} + \frac{x^2\,\ell n^2 a}{2!} + \frac{x^3\,\ell n^3 a}{3!} + ...a > 0$

(ii) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$

(iii) $\ell n\,(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\text{for} -1 < x \le 1$

(iv) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$

(v) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$

(vi) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + ...$

(vii) $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$

(viii) $\sin^{-1}x = x + \frac{1^2}{3!}x^3 + \frac{1^2.3^2}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + ...$

(ix) $\sec^{-1}x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + ...$

(x) $(1 + x)^n = 1 + \frac{nx}{\underline{|1}} + n(n-1)\frac{x^2}{\underline{|2}} + ...$ for $-1<x<1$.

Ex.29 Find $\lim\limits_{x\to 0}\frac{e^{\sin x} - \sin x - 1}{x^2}$

Sol. $\lim\limits_{x\to 0}\frac{e^{\sin x} - \sin x - 1}{x^2} = \lim\limits_{x\to 0}\frac{\left(1 + \frac{\sin x}{\underline{|1}} + \frac{\sin^2 x}{\underline{|2}} + \frac{\sin^3 x}{\underline{|3}} + ...\right) - \sin x - 1}{x^2}$

$= \lim\limits_{x\to 0}\frac{\left(1 + \frac{1}{\underline{|2}} + \frac{\sin x}{\underline{|3}} + ...\right)}{x^2} = \lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right)^2\left(\frac{1}{\underline{|2}} + \frac{\sin x}{\underline{|3}} + ...\right) = \frac{1}{\underline{|2}} = \frac{1}{2}$

Ex.30 Find $\lim\limits_{x=1}\frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$.

Sol. This is of the form $\frac{0}{0}$ if we put x = 1. Therefore we put x = 1 + h and expand.

$\lim\limits_{x=1}\frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2} = \lim\limits_{h=0}\frac{(1+h)^7 - 2(1+h)^5 + 1}{(1+h)^3 - 3(1+h)^2 + 2} = \lim\limits_{h=0}\frac{(1+7h+21h^2+...) - 2(1+5h+19h^2+...) + 1}{(1+3h+3h^2+...) - 3(1+2h+h^2) + 2}$

$= \lim\limits_{h=0}\frac{-3h + h^2 + ...}{-3h + ...} = \lim\limits_{h=0}\frac{-3 + h + ...}{-3 + ...} = 1.$

Ex.31 Evaluate $\operatorname{Limit}\limits_{x\to 0}\frac{3\sin x - \sin 3x}{x - \sin x}$

Sol. $\operatorname{Limit}\limits_{x\to 0}\frac{4\sin^3 x}{x - \sin x} = \operatorname{Limit}\limits_{x\to 0}\frac{4x^3}{x - \sin x} = \operatorname{Limit}\limits_{x\to 0}\frac{4x^3}{x - (x - \frac{x^3}{3!}....)} = 24$.

Ex.32 Let f(x) be a function such that $\lim_{x\to 0}\frac{f(x)}{x}$ = 1. Find the values of a and b such that

$$\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{\{f(x)\}^3}=1$$

Sol. Since, $\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{\{f(x)\}^3}=1 \Rightarrow \lim_{x\to 0}\frac{x\left(1+a\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}\right)\right)-b\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-....\right)}{\{f(x)\}^3}$ = 1

$$\Rightarrow \lim_{x\to 0}\frac{x(1+a-b)+x^3\left(\frac{a}{2!}+\frac{b}{3!}\right)+x^5\left(\frac{a}{4!}-\frac{b}{5!}\right)+...}{\{f(x)\}^3} = 1 \Rightarrow \lim_{x\to 0}\frac{\frac{(1+a-b)}{x^2}+\left(\frac{a}{2!}+\frac{b}{3!}\right)+x^2\left(\frac{a}{4!}-\frac{b}{5!}\right)+...}{\left\{\frac{f(x)}{x}\right\}^3} = 1$$

$\because$ R.H.S. is finite then L.H.S. is also finite, then 1 + a –b = 0 and $-\frac{a}{2!}+\frac{b}{3!}$ = 1

$\Rightarrow$ –3a + b = 6 then we get, a = –5/2 and b = –3/2

Ex.33 If the $\lim_{x\to 0}\frac{1}{x^3}\left(\frac{1}{\sqrt{1+x}}-\frac{1+ax}{1+bx}\right)$ exists and has the value equal to *l*, then find the value of $\frac{1}{a}-\frac{2}{l}+\frac{3}{b}$.

Sol. $\underset{x\to 0}{\text{Lt}}\frac{1}{x^3}\left[\frac{1}{\sqrt{1+x}}-\frac{1+ax}{1+bx}\right] = \underset{x\to 0}{\text{Lt}}\frac{1+bx-(1+ax)\sqrt{1+x}}{x^3} = \underbrace{\underset{x\to 0}{\text{Lt}}\frac{1}{\sqrt{1+x}\ (1+bx)}}_{=1}$

$= \underset{x\to 0}{\text{Lt}}\frac{1+bx-(1+ax)\{1+x\}^{1/2}}{x^3}$ Use binomial expansion to get the following relations :

$b-\frac{1}{2}-a=0$, $-\frac{1}{8}+\frac{a}{2}$ = 0, a = $\frac{1}{4}$ b = $\frac{3}{4}$ $\ell = -\frac{1}{32}$

(3) LIMIT OF EXPONENTIAL FUNCTIONS

$\lim_{x\to 0}\frac{a^x-1}{x}$ = *l*n a (a > 0). **In particular** $\lim_{x\to 0}\frac{e^x-1}{x}$ = 1

Ex.34 Evaluate $\lim_{h\to 0}\frac{(x+h)^{\sqrt{x+h}}-x^{\sqrt{x}}}{h}$ (x > 0).

Sol. $\lim_{h\to 0}\frac{e^{\sqrt{x+h}\,\ell n(x+h)}-e^{\sqrt{x}\,\ell nx}}{h} = \lim_{h\to 0} e^{\sqrt{x}\,\ell nx}\left[\frac{e^{\sqrt{x+h}\,\ell n(x+h)-\sqrt{x}\,\ell nx}-1}{h}\right]$

$$= x^{\sqrt{x}}\left[\lim_{s\to 0}\frac{e^s-1}{s}\cdot\lim_{h\to 0}\frac{\sqrt{x+h}\;\ell n\,(x+h)-\sqrt{x}\;\ell n\,x}{h}\right] \quad = x^{\sqrt{x}}\lim_{h\to 0}\frac{\sqrt{x+h}\left(\ell nx+\ell n\left(1+\frac{h}{x}\right)\right)-\sqrt{x}\,\ell nx}{h}$$

$$= x^{\sqrt{x}}\left[\lim_{h\to 0}\ell nx\frac{(\sqrt{x+h}-\sqrt{x})}{h}+\lim_{h\to 0}\frac{\sqrt{(x+h)}}{x}\;\ell n\left(1+\frac{h}{x}\right)^{x/h}\right] = x^{\sqrt{x}}\left[\frac{\ell nx}{2\sqrt{x}}+\frac{1}{\sqrt{x}}\right]$$

Ex.35 Evaluate $\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x}$.

Sol. Let $L = \lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x^3.\frac{x-\sin x}{x^3}}$

Now $\ell_1 = \lim_{x\to 0}\frac{x-\sin x}{x^3} = \frac{1}{6}$ (use x = 3 t)

$\ell_2 = \lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x^3}$ put $x = 3y$

$= \text{Limit}_{y\to 0}\frac{e^{3y}-e^{-3y}-6y}{27y^3} = \text{Limit}_{y\to 0}\frac{(e^y-e^{-y})^3+3(e^y-e^{-y})-6y}{27y^3}$

$\left[\text{Limit}_{y\to 0}\frac{8}{27}\frac{(e^{2y}-1)^3}{(2y)^8}+\frac{1}{9}\ell_2\right]$ $\ell_2 = \frac{8}{27}+\frac{\ell_2}{9} \Rightarrow \ell_2 = \frac{1}{3}$ Hence $L = \frac{\ell_1}{\ell_2} = 2$

(4) (a) $\lim_{x\to 0}(1+x)^{1/x} = e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ **Note :** $\lim_{x\to 0}\frac{1}{x}\,\ell n\,(1+x) = 1$

(b) Generalized formula for 1^∞ form :

If $\lim_{x\to a}f(x) = 1$ and $\lim_{x\to a}\phi(x) = \infty$, then $\lim_{x\to a}\left[f(x)\right]^{\phi(x)} = e^{\lim_{x\to a}\phi(x)[f(x)-1]}$

Ex.36 Evaluate $\lim_{t\to 0}\frac{\ell n\cos(\sin t)}{t^2}$

Sol. $\lim_{t\to 0}\frac{\ell n\cos(\sin t)}{t^2} = \lim_{t\to 0}\frac{\ell n\left(1-2\sin^2\frac{(\sin t)}{2}\right)}{-2\sin^2\left(\frac{\sin t}{2}\right)}.\frac{\left(-2\sin^2\left(\frac{\sin t}{2}\right)\right)}{t^2} \quad = \lim_{t\to 0}-2.\left(\frac{\sin\left(\frac{\sin t}{2}\right)}{\frac{\sin t}{2}}\right)^2.\frac{\sin^2 t}{4.t^2} = -\frac{1}{2}$

Ex.37 Find $\lim_{x\to\pi/4}\frac{\ln\tan x}{1-\cot x}$

Sol. Put $x = t + \pi/4 \Rightarrow \lim_{t\to 0}\frac{\ln\tan(t+\pi/4)}{1-\cot(t+\pi/4)} = \lim_{t\to 0}\frac{\ln\left[\frac{1+\tan t}{1-\tan t}\right]}{1-\left[\frac{\cot t\cot\pi/4-1}{\cot t+\cot\pi/4}\right]}$

$= \lim_{t\to 0}\frac{\ln(1+\tan t)}{\frac{2\tan t}{t+\tan t}} + \lim_{t\to 0}\frac{\ln(1-\tan t)(1+\tan t)}{-2\tan t} = \frac{1}{2}\,[1.1 + 1.1] = \frac{2}{2} = 1.$

Ex.38 Evaluate $\lim\limits_{x\to0}\left(\frac{\ell n(\cos 3x)}{x^2}.\frac{2\sin x}{e^x - e^{-x}}\right)$.

Sol. $\lim\limits_{x\to0}\frac{\ell n(\cos 3x).2\sin x}{x^2(e^x-e^{-x})} = \lim\limits_{x\to0}\frac{\ell n(\cos^2 3x).\sin x.e^x}{x^2(e^{2x}-1)}$

$$= \lim_{x\to0}\left(\frac{\ell n(1+\cos 3x-1)}{(\cos 3x-1)}\times(\cos 3x-1)\times\frac{2\sin x}{x}\times\frac{1}{\left(\frac{e^x-1}{x}+\frac{e^{-x}-1}{-x}\right)x^2}\right)$$

$$= \lim_{x\to0}\left(\frac{\ell n(1+\cos 3x-1)}{(\cos 3x-1)}\times\frac{\left(-2\sin^2\frac{3x}{2}\right)}{x^2}\times\frac{2\sin x}{x}\times\frac{1}{\left(\frac{e^x-1}{x}+\frac{e^{-x}-1}{-x}\right)}\right)$$

$$= 1 \times (-2) \times \frac{9}{4} \times 2 \times 1 \times \frac{1}{1+1} = -\frac{9}{2}.$$

Ex.39 Evaluate $\lim\limits_{x\to0}\frac{\ell n\left(1+x^2+x^4\right)}{\left(e^x-1\right).x}$.

Sol. $\lim\limits_{x\to0}\frac{\ell n\left(1+x^2+x^4\right)}{x^2} = \lim\limits_{x\to0}\ell n\ (1+x^2+x^4)^{\frac{1}{x^2}}$

$$= \ell n \lim_{x\to0} e^{\frac{1}{x^2}\left(1+x^2+x^4-1\right)} = \lim_{x\to0}\frac{x^2+x^4}{x^2} = \lim_{x\to0}(1+x^2) = 1$$

Ex.40 Evaluate : $\lim\limits_{x\to0}\left[\sin^2\left(\frac{\pi}{2-ax}\right)\right]^{\sec^2\left(\frac{\pi}{2-bx}\right)}$

Sol. $\ell = \lim\limits_{x\to0}\left[\sin^2\left(\frac{\pi}{2-ax}\right)\right]^{\sec^2\frac{\pi}{2-bx}} = e^{-\lim\limits_{x\to0}\sec^2\frac{\pi}{2-bx}\cos^2\frac{\pi}{2-ax}}$

consider $\lim\limits_{x\to0}\frac{\cos\frac{\pi}{2-ax}}{\cos\frac{\pi}{2-bx}} = \lim\limits_{x\to0}\frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{2-ax}\right)}{\sin\left(\frac{\pi}{2}-\frac{\pi}{2-bx}\right)}$

$$= \lim_{x\to0}\frac{\frac{\pi}{2}-\frac{\pi}{2-ax}}{\frac{\pi}{2}-\frac{\pi}{2-bx}} = \lim_{x\to0}\frac{-ax}{2-ax}\ \frac{2-bx}{-bx} = \frac{a}{b} \qquad \therefore\ \ell = e^{-\frac{a^2}{b^2}}$$

Ex.41 Evaluate $\lim_{x\to 0}\left(\frac{(1+[x])^{1/\{x\}}}{e}\right)^{1/\{x\}}$ if it exist (where {x} denotes the fractional part of x)

Sol. L.H.L. $= \lim_{x\to 0-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0}\left(\frac{1+\{0-h\})^{1/\{0-h\}}}{e}\right)^{1/\{0-h\}} = \lim_{h\to 0}\left(\frac{(1+1-h)^{1/(1-h)}}{e}\right)^{1/(1-h)} = \left(\frac{2}{e}\right)$

R.H.L. $= \lim_{x\to 0+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0}\left(\frac{(1+\{h\})^{1/\{h\}}}{e}\right)^{1/\{h\}} = \lim_{h\to 0}\left(\frac{(1+h)^{1/h}}{e}\right)^{1/h}$

$= e^{\frac{1}{e}\lim_{h\to 0}}\ \frac{((1+h)^{1/h}-e)}{h} = e^{\frac{1}{e}\lim_{h\to 0}}\left(\frac{e^{(1/h)\ln(1+h)}-e}{h}\right) = e^{\lim_{h\to 0}}\left[\frac{e^{\frac{\ln(1+h)-h}{h}}-1}{h}\right]$

$= e^{\lim_{h\to 0}\left(\frac{(e^{\frac{\ln(1+h)}{h}}-1)}{\frac{\ln(1+h)-h}{h}}\right)\cdot\left(\frac{\ln(1+h)-h}{h^2}\right)}$ ...(1)Now let $P = \lim_{h\to 0}\frac{\ln(1-h)-h}{h^2}$ ...(2)

replacing h by –h then $P = \lim_{h\to 0}\frac{\ln(1-h)+h}{h^2}$ ...(3)

adding (2) and (3), $2P = \lim_{h\to 0}\frac{\ln(1-h^2)}{h^2} = -\lim_{h\to 0}\frac{\ln(1-h^2)}{(-h^2)} = -1$

$\therefore\ P = -\frac{1}{2}$ from (1), R.H.L. $= e^{1.(-1/2)} = e^{-1/2}$ $\therefore$ L.H.L. ≠ R.H.L. Hence P does not exist.

Ex.42 Evaluate $\lim_{x\to\infty} (\pi - 2\tan^{-1} x) \ln x$

Sol. $\lim_{x\to\infty}\frac{(\pi-2\tan^{-1}x)}{\frac{1}{\ell nx}}$ $\left(\frac{0}{0}\right)$ Applying L' Hospital's Rule$= \lim_{x\to\infty}\frac{-\frac{2}{1+x^2}}{\frac{1}{(\ln x)^2}\times\frac{1}{x}} = \lim_{x\to\infty}\frac{2}{1+x^2}\ (\ln x)^2$

$= \lim_{x\to\infty}\frac{2(\ln x)^2+2x.2\ln x.\frac{1}{x}}{2x} = \lim_{x\to\infty}\frac{(\ln x)^2+2\ln x}{x} = \lim_{x\to\infty}\frac{2\ln x.\frac{1}{x}+\frac{2}{x}}{x}$

$= \lim_{x\to\infty}\frac{1}{x^2}\ [2\ln x + 3] = 0$ $\quad \frac{1}{x}\to 0$

Ex.43 Find a polynomial of least degree, such that $\lim\limits_{x\to 0}\left(1+\dfrac{x^2+f(x)}{x^2}\right)^{1/x} = e^2$.

Sol. Now, $\lim\limits_{x\to 0}\left(1+\dfrac{x^2+f(x)}{x^2}\right)^{1/x}$ exists only when $\lim\limits_{x\to 0}\dfrac{x^2+f(x)}{x^2} = 0$ (i.e. it converts to 1^∞ form).

So, the least degree in f(x) is of degree 2. i.e. $f(x) = a_2x^2 + a_3x^3 + ...$

Now, $\lim\limits_{x\to 0}\left(1+\dfrac{x^2+f(x)}{x^2}\right)^{1/x} = e^2 \Rightarrow e^{\lim\limits_{x\to 0}\left(\frac{x^2+f(x)}{x^2}\right)\frac{1}{x}} = e^2 \Rightarrow e^{\lim\limits_{x\to 0}\frac{x^2+f(x)}{x^3}} = e^2$

$\therefore \quad \lim\limits_{x}\dfrac{x^2+f(x)}{x^3} = 2 \quad \Rightarrow \quad \lim\limits_{x}\dfrac{x^2+a_2x^2+a_3x^3+.....}{x^3} = 2$

$\Rightarrow a_2 = -1$, $a_3 = 2$ and a_4, a_5 .... are any arbitrary constants. Since, we want polynomial of least degree.

Hence, $f(x) = x^2 + 2x^3$.

Ex.44 Evaluate $\underset{x\to\infty}{\text{Lim}}\ x - x^2\,\ell n\left(1+\dfrac{1}{x}\right)$.

Sol. Put $x = \dfrac{1}{y} \quad \Rightarrow \quad$ as $x\to\infty$; $y\to 0$.

Hence $\ell = \underset{y\to 0}{\text{Lim}}\ \dfrac{1}{y}-\dfrac{1}{y^2}\ \ell n\ (1+y) \quad = \underset{y\to 0}{\text{Lim}}\ \dfrac{y-\ell n(1+y)}{y^2}$

Put $1 + y = e^z$ as $y\to 0$, $z\to 0$

$= \underset{z\to 0}{\text{Lim}}\ \dfrac{(e^z-1)-z}{(e^z-1)^2} \quad = \underset{z\to 0}{\text{Lim}}\ \dfrac{e^z-z-1}{z^2}$ Put $z = 2t$

$= \underset{t\to 0}{\text{Lim}}\ \dfrac{e^{2t}-2t-1}{4t^2} \quad = \underset{t\to 0}{\text{Lim}}\ \dfrac{(e^t-1)^2+2e^t-2t-2}{4t^2}$

$\ell = \dfrac{1}{4} + \dfrac{l}{2} \quad \Rightarrow \quad \ell = \dfrac{1}{2}$

(5) BINOMIAL LIMIT : $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=na^{n-1}$

Ex.45 Let $\lim\limits_{x\to a}\dfrac{x^x-a^x}{x-a}=\ell$, $a>0$ & $\lim\limits_{x\to a}\dfrac{a^x-x^a}{x-a}=m$, $a>0$. If $\ell=m$ then find the value of 'a'.

Sol. $\ell=\lim\limits_{x\to a}\dfrac{e^{x\ell n\,x}-e^{x\ell n\,a}}{x-a}=\dfrac{e^{x\ell n\,a}\left[e^{x(\ell n\,x-\ell n\,a)}-1\right]}{x(\ell n\,x-\ell n\,a)}\cdot\dfrac{x(\ell n\,x-\ell n\,a)}{x-a}$

$$=a^a\cdot\lim_{h\to 0}\frac{(a+h)\,\ell n\left(1+\dfrac{h}{a}\right)}{h}=a^a$$

$m=\lim\limits_{x\to a}\dfrac{e^{x\ell n\,a}-e^{a\ell n\,x}}{x-a}=\dfrac{e^{a\ell n\,x}\left[e^{x\ell n\,a-a\ell n\,x}-1\right]}{\lim\limits_{x\to a}x\ell n\,a-a\ell n\,x}\cdot\dfrac{x\ell n\,a-a\ell n\,x}{x-a}$

$$=a^a\cdot\lim_{h\to 0}\frac{(a+h)\,\ell n\,a-a\ell n\,(a+h)}{h}=a^a\lim_{h\to 0}\left[\ell n\,a-\ell n\left(1+\frac{h}{a}\right)^{a/h}\right]=a^a\,(\ell n\,a-1)$$

Now $\ell=m \Rightarrow a=e^2$

Ex.46 Evaluate $\underset{n\to\infty}{\text{Limit}}\left[\tan\dfrac{x}{2}.\sec x+\tan\dfrac{x}{2^2}.\sec\dfrac{x}{2}+\tan\dfrac{x}{2^3}.\sec\dfrac{x}{2^2}+......+\tan\dfrac{x}{2^n}.\sec\dfrac{x}{2^{n-1}}\right]$

where $x\in\left(0,\dfrac{\pi}{2}\right)$.

Sol. $T_1=\dfrac{\sin\dfrac{x}{2}}{\cos\dfrac{x}{2}\cos x}=\dfrac{\sin\left(x-\dfrac{x}{2}\right)}{\cos\dfrac{x}{2}\cos x}=\tan x-\tan\dfrac{x}{2}$

Similarly $T_2=\tan\dfrac{x}{2}-\tan\dfrac{x}{2^2}$ and $T_3=\tan\dfrac{x}{2^2}-\tan\dfrac{x}{2^3}$

$T_n=\tan\dfrac{x}{2^{n-1}}-\tan\dfrac{x}{2^n}$ $\quad S=\tan x-\tan\dfrac{x}{2^n}$ $\quad\therefore \lim\limits_{n\to\infty}S=\tan x$

Ex.47 Define $f(n,\theta) = \left(1-\tan^2\frac{\theta}{2}\right)\left(1-\tan^2\frac{\theta}{2^2}\right)\left(1-\tan^2\frac{\theta}{2^3}\right)$ ...... to n factors.

Show that $\lim\limits_{n\to\infty} f(n,\theta) = \frac{\theta}{\tan\theta}$.

Sol. $1-\tan^2\frac{\theta}{2} = \frac{\cos\theta}{\cos^2\frac{\theta}{2}}$; $1-\tan^2\frac{\theta}{2^2} = \frac{\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2^2}}$ and so on .

Hence $f(n,\theta) = \cos\theta \cdot \frac{1}{\cos\frac{\theta}{2}}\cdot\frac{1}{\cos\frac{\theta}{2^2}}\cdots\cdots\frac{1}{\cos\frac{\theta}{2^{n-1}}}\cdot\frac{1}{\cos^2\frac{\theta}{2^n}}$

But $\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2^2}\cdot\cos\frac{\theta}{2^3}\ \ldots\ldots\ \cos\frac{\theta}{2^n} = \frac{\sin\theta}{2^n\sin\frac{\theta}{2^n}}$

Hence $f(n,\theta) = \cot\theta \cdot 2^n\tan\frac{\theta}{2^n} = \cot\theta \cdot \frac{\theta\tan\frac{\theta}{2^n}}{\left(\frac{\theta}{2^n}\right)} \to \theta\cot\theta$ as $n\to\infty$

Ex.48 Let $f(x) = \lim\limits_{n\to\infty}\sum_{n=1}^{n} 3^{n-1}\sin^3\frac{x}{3^n}$ and $g(x) = x - 4f(x)$. Evaluate $\lim\limits_{x\to 0}(1+g(x))^{\cot x}$.

Sol. Using $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$T_1 = \sin^3\frac{x}{3} = \frac{1}{4}\left(3\sin\frac{x}{3}-\sin x\right)$ $T_2 = 3\sin^3\frac{x}{3^2} = \frac{3}{4}\left(3\sin\frac{x}{3^2}-\sin\frac{x}{3}\right)$ and so on........

$\therefore\ T_1 = \frac{1}{4}\left(3\sin\frac{x}{3}-\sin x\right)$ $T_2 = \frac{1}{4}\left(3^2\sin\frac{x}{3^2}-3\sin\frac{x}{3}\right)$ $T_n = \frac{1}{4}\left(3^n\sin\frac{x}{3^n}-3^{n-1}\sin\frac{x}{3^{n-1}}\right)$

$\therefore\ f(x) = \lim\limits_{n\to\infty}\frac{1}{4}\left(3^n\sin\frac{x}{3^n}-\sin x\right) = \frac{1}{4}\left(\lim\limits_{n\to\infty}\frac{x\sin\frac{x}{3^n}}{\frac{x}{3^n}}-\sin x\right) = \frac{1}{4}(x-\sin x)$

$g(x) = x - 4f(x) = \sin x$

now $\lim\limits_{x\to 0}(1+\sin x)^{\cot x} = e^{\lim\limits_{x\to 0}(\cot x)(\sin x)} = e$

Ex.49 Evaluate $\lim\limits_{n\to\infty}\sum_{r=1}^{n}\cot^{-1}\left(\frac{r^3-r+\frac{1}{r}}{2}\right)$

Sol. Here, $\lim\limits_{n\to\infty}\sum_{r=1}^{n}\cot^{-1}\left(\frac{r^3-r+\frac{1}{r}}{2}\right)$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\tan^{-1}\left(\frac{2r}{1-r^2+r^4}\right) \qquad = \lim_{n\to\infty}\sum_{r=1}^{n}\tan^{-1}\left(\frac{2r}{1+(r^2-r)(r^2+r)}\right)$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\tan^{-1}\left(\frac{(r^2+r)(r^2-r)}{1+(r^2+r)(r^2-r)}\right) = \lim_{n\to\infty}\sum_{r=1}^{n}\{\tan^{-1}(r^2+r)-\tan^{-1}(r^2-r)\}$$

$$= \lim_{n\to\infty} [(\tan^{-1}2 - \tan^{-1}0) + (\tan^{-1}6 - \tan^{-1}2) + (\tan^{-1}12 - \tan^{-1}6)$$

$$= \lim_{n\to\infty} \{\tan^{-1}(n^2+n) - \tan^{-1}(0)\} \qquad = \tan^{-1}(\infty) - \tan^{-1}(0) = \frac{\pi}{2}$$

$$\Rightarrow \lim_{n\to\infty}\sum_{r=1}^{n}\cot^{-1}\left(\frac{r^3-r+\frac{1}{r}}{2}\right) = \frac{\pi}{2}$$

Ex.50 Evaluate $\lim_{n\to\infty}\left(\frac{a_1+1}{a_1}\right)\left(\frac{a_2+1}{a_2}\right)........\left(\frac{a_n+1}{a_n}\right)$, where $a_1 = 1$ and $a_n = n(1 + a_{n-1})\ \forall\ n \ge 2$.

Sol. $\lim_{n\to\infty}\left(\frac{a_1+1}{a_1}\right)\left(\frac{a_2+1}{a_2}\right)......\left(\frac{a^n+1}{a_n}\right)$ We know, $a_{n-1} + 1 = \frac{a_n}{n}$ ...(i)

$$\lim_{n\to\infty}\left(\frac{a_2}{2}\right)\left(\frac{a_3}{3}\right)\left(\frac{a_4}{4}\right).....\left(\frac{a_{n+1}}{n+1}\right)\frac{1}{a_1.a_2.....a_n} \qquad = \lim_{n\to\infty}\frac{a_{n+1}}{(n+1)!} = \lim_{n\to\infty}\frac{1+a_n}{n!} \qquad \{\text{using (ii)}\}$$

$$= \lim_{n\to\infty}\left(\frac{1}{n!}+\frac{a_n}{n!}\right) \qquad = \lim_{n\to\infty}\left(\frac{1}{n!}+\frac{1}{(n-1)!}+\frac{a_{n-1}}{(n-1)!}\right) \qquad \{\text{using (i)}\}$$

$$= \lim_{n\to\infty}\left(\frac{1}{n!}+\frac{1}{(n-1)!}+\frac{1}{(n-2)!}+......+\frac{1}{(2)!}+\frac{1}{1!}+\frac{a_1}{1!}\right) \qquad \{a_1 = 1;\ \text{given}\}$$

$$= \lim_{n\to\infty}\left(\frac{1}{n!}+\frac{1}{(n-1)!}+\frac{1}{(n-2)!}+......+\frac{1}{(2)!}+\frac{1}{1!}+\frac{1}{1}\right)$$

$$= e \qquad \left\{\text{as, } e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+........\infty\right\}$$

Ex.51 The function u_n takes on the following values :

$u_1 = \frac{1}{4}$, $u_2 = \frac{1}{4} + \frac{1}{10}$ .....$u_n = \frac{1}{3+1} + \frac{1}{3^2+1} + ...\frac{1}{3^n+1}$ .... Prove that $\lim_{n\to\infty} u_n < \frac{1}{2}$

Sol. Let $u_1 = \frac{1}{3}$ $u_2 = \frac{1}{9}$ $u_3 = \frac{1}{27}$ .... $u_n = \frac{1}{3^n}$ $\frac{1}{4} < \frac{1}{3}$; $\frac{1}{4} + \frac{1}{10} < \frac{1}{3} + \frac{1}{10}$

$$u_n < \frac{1}{3} + \frac{1}{3^2} + ...\frac{1}{3^n} < \frac{\frac{1}{3}\left(1-\left(\frac{1}{3}\right)^n\right)}{1-\frac{1}{3}} < \frac{1}{3}\left(1-\frac{1}{3}\right)^n . \frac{3}{2} < \frac{1}{2}\left(1-\left(\frac{1}{3}\right)^n\right)$$

$$\lim_{n\to\infty} u_n < \lim_{n\to\infty} \frac{1}{2}\left(1-\left(\frac{1}{3}\right)^n\right) = \frac{1}{2}.$$

Ex.52 Solve $\lim_{n\to\infty} x_n$, when $x_n^2 = a + x_{n-1}$ and $x_0 = \sqrt{a}$.

Sol. Here, $x_0 = \sqrt{a}$ $x_1 = \sqrt{a+\sqrt{a}}$ which shows $x_1 > x_0$

$\therefore$ assuming $x_n > x_{n-1}$ $\Rightarrow$ $x_{n-1}^2 - a < 0$

$$\Rightarrow \left\{x_{n-1} - \frac{\sqrt{4a+1}+1}{2}\right\}\left\{x_{n-1} + \frac{\sqrt{4a+1}-1}{2}\right\} < 0$$

$\Rightarrow$ $x_{n-1} < \frac{\sqrt{4a+1}+1}{2}$ $\therefore$ $\lim_{n\to\infty} x_{n-1} = \lim_{n\to\infty} x_n = \ell$

$\Rightarrow$ $\ell^2 - \ell - a = 0$ $\Rightarrow$ $\ell = \frac{1 \pm \sqrt{1+4a}}{2}$, since $\ell \geq 0$

$\therefore$ $\ell = \frac{1+\sqrt{1+4a}}{2}$ or $\lim_{n\to\infty} x_n < \frac{\sqrt{1+4a}+1}{2}$

Ex.53 The function u_n attains the values

$u_1 = \frac{1}{2}$; $u_2 = \frac{1}{2} + \frac{1}{2.4} + \frac{1}{2.4.6}$ $u_n = \frac{1}{2} + \frac{1}{2.4} + ... \frac{1}{2.4....(2n)}$,..... Prove that $\lim_{n\to\infty} u_n < 2$.

Sol. $\frac{1}{2} < 1$ $\frac{1}{2.4} < \frac{1}{2^2}$ $\frac{1}{2.4.6} < \frac{1}{2^3}$

$$\frac{1}{2} + \frac{1}{2.4} + \frac{1}{2.4.6} + ...\frac{1}{2.4...(2n)} < 1 + \frac{1}{2^2} + \frac{1}{2^3} + ... < \frac{\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}} \qquad \lim_{n\to\infty} u_n = \lim_{n\to\infty} 2\left(1-\left(\frac{1}{2}\right)^n\right) < 2.$$

Ex.54 If $a_1 = \sqrt{ab_1}$, $b_1 = \frac{a+b}{2}$, $a_2 = \sqrt{a_1b_2}$, $b_2 = \frac{a_1+b_1}{2}$ ... and so on for (a > b > 0) then prove that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \frac{\sqrt{a^2-b^2}}{\tan^{-1}\left(\frac{\sqrt{a^2-b^2}}{b}\right)}.$$

Sol. Put b = a cos θ we get $b_1 = a\cos^2\frac{\theta}{2}$ and $a_1 = a\cos\frac{\theta}{2}$

$b_2 = a\cos\frac{\theta}{2}\cos^2\frac{\theta}{4}$ and $a_2 = a\cos\frac{\theta}{2}\cos\frac{\theta}{4}$

similarly we get $a_n = a\cos\frac{\theta}{2}\cos^2\frac{\theta}{4}\ ......\ \cos\frac{\theta}{2^n}$ and $b_n = a\cos\frac{\theta}{2}\cos\frac{\theta}{4}\ ....\ \cos^2\frac{\theta}{2^2}$

$\Rightarrow\ a_n = \frac{a\sin\theta}{2^n\sin\theta/2^n}$ and $b_n = \frac{a\sin\theta\cos\theta/2^n}{2^n\sin\theta/2^n}$

Now, $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{a\sin\theta}{2^n\sin\theta/2^n} = \frac{a\sin\theta}{\theta}$ $\quad\Rightarrow\quad$ $\lim_{n\to\infty} a_n = \frac{a\sqrt{(a^2-b^2)}}{a\cos^{-1}(b/a)} = \frac{\sqrt{a^2-b^2}}{\tan^{-1}\left(\frac{\sqrt{a^2-b^2}}{b}\right)}$

and $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{a\sin\theta\cos\theta/2^n}{2^n\sin\theta/2^n} = \frac{a\sin\theta}{\theta}$ $\quad\Rightarrow\quad$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \frac{\sqrt{a^2-b^2}}{\tan^{-1}\left(\frac{\sqrt{a^2-b^2}}{b}\right)}$

Ex.55 Let $a_1, a_2, ..., a_n$ be sequence of real numbers with $a_{n+1} = a_n + \sqrt{1+a_n^2}$ and $a_0 = 0$.

Prove that $\lim_{n\to\infty}\left(\frac{a_n}{2^{n-1}}\right) = \frac{4}{\pi}$

Sol. Here, $a_{n+1} = a_n + \sqrt{1+a_n^2}$, where let a_n cot (α_n)

$\Rightarrow\ a_{n+1} = \cot(\alpha_n) + \text{cosec}(\alpha_n)$

$\Rightarrow\ a_{n+1} = \frac{\cos(\alpha_n)+1}{\sin(\alpha_n)} = \frac{2\cos^2(\alpha_n/2)}{2\sin(\alpha_n/2)\cos(\alpha_n/2)} = \cot\left(\frac{\alpha_n}{2}\right)$

Putting n = 1, $\qquad a_1 = \cot(\alpha_1)$ and $a_1 = a_0 + \sqrt{1+a_0^2} = 1$

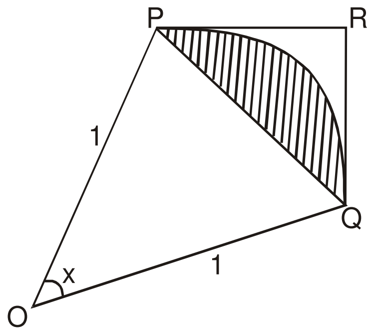
$\Rightarrow\ \cot(\alpha_1) = 1$ or $\alpha_1 = \frac{\pi}{4}$

again, $\quad a_2 = \cot\left(\frac{\alpha_1}{2}\right) = \cot\left(\frac{\pi}{8}\right) \qquad a_3 = \cot\left(\frac{\alpha_2}{2}\right) = \cot\left(\frac{\pi}{4.2^2}\right)$

$a_4 = \cot\left(\frac{\alpha_3}{2}\right) = \cot\left(\frac{\pi}{4.2^3}\right)$ .... $a_n \cot\left(\frac{\pi}{4.2^{n-1}}\right)$; Put $\frac{1}{2^{n-1}} = x$

Hence, $\lim_{n\to\infty}\left(\frac{a_n}{2^{n-1}}\right) = \lim_{n\to\infty}\frac{\cot\left(\frac{\pi}{4.2^{n-1}}\right)}{2^{n-1}} = \lim_{n\to\infty}\frac{1}{\frac{\tan\left(\frac{\pi}{4}x\right)}{x}}$ $\therefore$ $\lim_{n\to\infty}\left(\frac{a_n}{2^{n-1}}\right) = \frac{4}{\pi}$

H. GEOMETRICAL LIMITS



Ex.56 A circular arc of radius 1 subtends an angle of x radians, $0 < x < \pi/2$ as shown in the figure. The point R is the intersection of the two tangent line at P and Q. Let T(x) be the area of triangle PQR and let S(x) be the area of the shaded region then find

(i) T(x) **(ii)** S (x) **(iii)** $\lim_{x\to 0}\frac{T(x)}{S(x)}$

Sol. **(i)** In Δ OPR, $\tan\left(\frac{x}{2}\right) = \frac{PR}{1}$

$\therefore$ $PR = \tan\frac{x}{2} = RQ$ ($\because$ Length of tangent fram a point outside the circle are equal)

and $\angle PRQ = (\pi - x)$

$\therefore$ $T(x) = $ Area of Δ PQR $= \frac{1}{2}.(PR)(RQ)\sin(\pi - x) = \frac{1}{2}.\tan^2\left(\frac{x}{2}\right)\sin x$ ...(1)

$= \tan\left(\frac{x}{2}\right) - \frac{\sin x}{2}$

(ii) S(x) = area of Sector OPQ area od Δ OPQ

$= \frac{1}{2}(1)^2 . x - \frac{1}{2}.(1)^2 . \sin x = \frac{(x - \sin x)}{2}$

(iii) $\lim_{x\to 0}\frac{T(x)}{S(x)} = \lim_{x\to 0}\frac{\tan\left(\frac{x}{2}\right) - \frac{\sin x}{2}}{\frac{(x-\sin x)}{2}} = \lim_{x\to 0}\frac{\tan^2(x/2)\sin x}{(x - \sin x)}$

$= \lim_{x\to 0}\frac{\left(\frac{\tan(x/2)}{x/2}\right)^2\left(\frac{\sin x}{x}\right)}{(x-\sin x)}.\frac{x^3}{4} = \frac{1}{4}\lim_{x\to 0}\left(\frac{\tan(x/2)}{x/2}\right)^2 \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\lim_{x\to 0}\frac{x^3}{(x-\sin x)} = \frac{3}{2}.$

EXERCISE – I SINGLE CORRECT (OBJECTIVE QUESTIONS)

1. $\underset{x\to 1}{\text{Limit}}\,(1 - x + [x - 1] + [1 - x])$ is
(where [∗] denotes greatest integer function)
(A) 0 (B) 1 (C) –1 (D) does not exist
Sol.

2. $\lim\limits_{x\to\infty} \sec^{-1}\left(\frac{x}{x+1}\right) =$

(A) 0 (B) π (C) $\frac{\pi}{2}$ (D) does not exist
Sol.

3. $\underset{x\to 2}{\text{Limit}}\,\frac{\sin(e^{x-2}-1)}{\ln(x-1)} =$
(A) 0 (B) –1 (C) 2 (D) 1
Sol.

4. $\underset{x\to -\infty}{\text{Lim}}\,\frac{x^2 \sin\left(\frac{1}{x}\right)}{\sqrt{9x^2+x+1}}$, is
(A) $\frac{1}{3}$ (B) $-\frac{1}{3}$ (C) 0 (D) not exist
Sol.

5. The value of $\underset{x\to 0}{\text{Limit}}\,\frac{\sin(\ln(1+x))}{\ln(1+\sin x)}$ is
(A) 0 (B) 1/2 (C) 1/4 (D) 1
Sol.

6. $\underset{x\to\pi/2}{\text{Limit}}\left[\frac{x-\frac{\pi}{2}}{\cos x}\right]$ is (where [∗] denotes greatest integer function)
(A) –1 (B) 0 (C) –2 (D) does not exist
Sol.

7. The value of $\underset{x\to\pi/2}{\text{Limit}}$
$\tan^2 x\,\left(\sqrt{2\sin^2 x+3\sin x+4}-\sqrt{\sin^2 x+6\sin x+2}\right)$ is equal to
(A) 1/10 (B) 1/11 (C) 1/12 (D) 1/8
Sol.

8. $\underset{x\to\infty}{\text{Limit}}\,\frac{x^3.\sin\frac{1}{x}+x+1}{x^2+x+1} =$
(A) 0 (B) 1/2 (C) 1 (D) None of these
Sol.

9. $\underset{n\to\infty}{\text{Limit}}\dfrac{-3n+(-1)^n}{4n-(-1)^n}$ is

(A) $-\dfrac{3}{4}$ (B) $-\dfrac{3}{4}$ if n is even; $\dfrac{3}{4}$ if n is odd

(C) not exist if n is even ; $-\dfrac{3}{4}$ if n is odd

(D) + 1 if n is even ; does not exist if n is odd

Sol.

10. The limit $\lim\limits_{\theta\to 0}\left(\left[\dfrac{n\sin\theta}{\theta}\right]+\left[\dfrac{n\tan\theta}{\theta}\right]\right)$, $n \in N$ is

(where [$*$] denotes greatest integer function)

(A) 2n (B) 2n+1 (C) 2n−1 (D) does not exist

Sol.

11. $\underset{n\to\infty}{\text{Limit}}$ n cos $\left(\dfrac{\pi}{4n}\right)\sin\left(\dfrac{\pi}{4n}\right)$ has the value equal to

(A) $\pi/3$ (B) $\pi/4$ (C) $\pi/6$ (D) None of these

Sol.

12. $\lim\limits_{x\to 0}\left[\dfrac{\sin[x-3]}{[x-3]}\right]$

(where [$*$] denotes greatest integer function)

(A) 0 (B) 1 (C) does not exist (D) sin 1

Sol.

13. $\underset{x\to\infty}{\text{Limit}}\left(\dfrac{x+2}{x-2}\right)^{x+1} =$

(A) e^4 (B) e^{-4} (C) e^2 (D) None of these

Sol.

14. $\underset{x\to 0^+}{\text{Limit}}$ $(1 + \tan^2\sqrt{x})^{5/x} =$

(A) e^5 (B) e^2 (C) e (D) None of these

Sol.

15. The value of $\underset{x\to\pi/4}{\text{Limit}}$ $(1 + [x])^{1/\ln(\tan x)}$ is equal to

(where [$*$] denotes greatest integer function)

(A) 0 (B) 1 (C) e (D) e^{-1}

Sol.

16. $\underset{x\to\infty}{\text{Limit}}\left(\dfrac{x^2-2x+1}{x^2-4x+2}\right)^{x} =$

(A) 1 (B) 2 (C) e^2 (D) e

Sol.

17. The limit $\underset{x\to a}{\text{Limit}}\left(2-\dfrac{a}{x}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$ is equal to

(A) $e^{-a/\pi}$ (B) $e^{-2a/\pi}$ (C) $e^{-2/\pi}$ (D) 1

Sol.

18. If [x] denotes the greatest integer ≤ x, then $\underset{n\to\infty}{\text{Limit}}\frac{1}{n^4}$ $([1^3 x] + [2^3 x] + ... + [n^3 x])$ equals

(A) x/2 (B) x/3 (C) x/6 (D) x/4

Sol.

19. $\underset{n\to\infty}{\text{Limit}}\frac{5^{n+1}+3^n-2^{2n}}{5^n+2^n+3^{2n+3}}$ =

(A) 5 (B) 3 (C) 1 (D) zero

Sol.

20. If f(x) = $\begin{cases} x-1, & x\ge 1 \\ 2x^2-2, & x<1 \end{cases}$, g(x) = $\begin{cases} x+1 & x>0 \\ -x^2+1, & x\le 0 \end{cases}$

and h(x) = |x| then find $\underset{x\to 0}{\text{Lim}}$ f(g(h(x)))

(A) 1 (B) 0

(C) –1 (D) does not exists

Sol.

21. $\underset{x\to\pi/2}{\text{Limit}}\frac{\left(1-\tan\frac{x}{2}\right)(1-\sin x)}{\left(1+\tan\frac{x}{2}\right)(\pi-2x)^3}$ is

(A) 1/16 (B) –1/16 (C) 1/32 (D) –1/32

Sol.

22. Let f(x) = $\begin{cases} x\sin\left(\frac{1}{x}\right)+\sin\left(\frac{1}{x^2}\right) & x\neq 0 \\ 0 & x=0 \end{cases}$, then $\lim\limits_{x\to\infty}$ f(x) equals

(A) 0 (B) –1/2 (C) 1 (D) None of these

Sol.

23. $\underset{x\to 0^+}{\text{Limit}}\frac{\cos^{-1}(1-x)}{\sqrt{x}}$ =

(A) $\frac{1}{\sqrt{2}}$ (B) $\sqrt{2}$ (C) 1 (D) 0

Sol.

24. If α and β be the roots of $ax^2 + bx + c = 0$, then

$\lim\limits_{x\to\alpha}(1+ax^2+bx+c)^{\frac{1}{x-\alpha}}$ is

(A) a (α – β) (B) ln |a (α – β)|

(C) $e^{a(\alpha-\beta)}$ (D) $e^{a|\alpha-\beta|}$

Sol.

25. $\underset{x\to\pi/2}{\text{Lim}}\frac{2^{-\cos x}-1}{x(x-\pi/2)}$ =

(A) $\frac{2\ell n2}{\pi}$ (B) $\ell n2$

(C) $\frac{2}{\pi}$ (D) does not exists

Sol.

26. $\lim\limits_{x\to 0} (\cos mx)^{n/x^2}$

(A) $e^{-m^2n/4}$ (B) $e^{-m^2n/2}$ (C) $e^{-mn^2/2}$ (D) $e^{-mn^2/4}$

Sol.

27. $\lim\limits_{x\to 0}\left[(1-e^x)\dfrac{\sin x}{|x|}\right]$ is

(where [∗] denotes greatest integer function)

(A) –1 (B) 1

(C) 0 (D) does not exist

Sol.

28. $\text{Limit}_{x\to 0}\dfrac{(4^x-1)^3}{\sin\left(\dfrac{x}{p}\right)\ln\left(1+\dfrac{x^2}{3}\right)} =$

(A) 9 p (log 4) (B) 3 p $(\log 4)^3$

(C) 12 p $(\log 4)^3$ (D) 27 p $(\log 4)^2$

Sol.

29. Given a real valued function f such that

$$f(x) = \begin{cases} \dfrac{\tan^2[x]}{(x^2-[x]^2)} & x>0 \\ 1 & x=0 \\ \sqrt{\{x\}\cot\{x\}} & x<0 \end{cases} \text{ then}$$

(where [∗] denotes greatest integer function and { ∗ } denotes fractional part function)

(A) $\lim\limits_{x\to 0} f(x) = 1$ (B) $\lim\limits_{x\to 0^-} f(x) = \cot 1$

(C) $\cot^{-1}\left(\lim\limits_{x\to 0^-} f(x)\right)^2 = 1$

(D) f is continuous at x = 0

Sol.

30. The limit $\lim\limits_{x\to 0}\dfrac{1-\cos x\sqrt{\cos 2x}}{x^2}$ is equal to

(A) 1/2 (B) –1/2 (C) 3/2 (D) 1

Sol.

31. $\text{Limit}_{x\to\infty}\ x - x^2 \ln\left(1+\dfrac{1}{x}\right)$ is equal to

(A) 1/2 (B) 3/2 (C) 1/3 (D) 1

Sol.

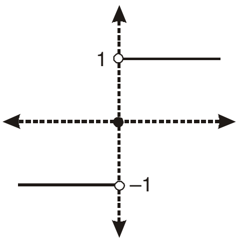
32. Evaluate $\text{Limit}_{n\to\infty}\left(\dfrac{1}{\sqrt{n^2}}+\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+....+\dfrac{1}{\sqrt{n^2+2n}}\right)$

(A) 1 (B) 1/2 (C) 0 (D) 2

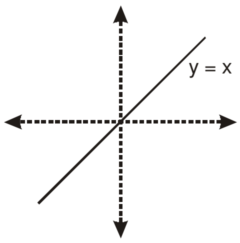
Sol.

33. The graph of the function $f(x)=\lim\limits_{t\to 0}\left(\dfrac{2x}{\pi}\cot^{-1}\dfrac{x}{t^2}\right)$, is

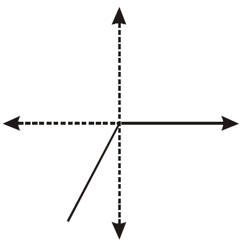
(A)



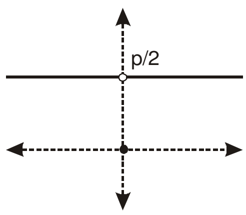
(B)



(C)



(D)



Sol.

34. If $\lim\limits_{x\to\infty} f(x)$ exist and is finite & non zero and if $\lim\limits_{x\to\infty}\left(f(x)+\dfrac{3f(x)-1}{f^2(x)}\right) = 3$ then the value of $\lim\limits_{x\to\infty} f(x)$ is

(A) 1 (B) –1

(C) 2 (D) none of these

Sol.

35. The value of $\text{Limit}_{x\to 0}\dfrac{\cos(\sin x)-\cos x}{x^4}$ is equal to

(A) 1/5 (B) 1/6 (C) 1/4 (D) 1/2

Sol.

36. $\text{Limit}_{x\to\infty}\dfrac{e^x\left(\left(2^{x^n}\right)^{\frac{1}{e^x}}-\left(3^{x^n}\right)^{\frac{1}{e^x}}\right)}{x^n}$, $n \in N$ is equal to

(A) 0 (B) ln (2/3) (C) ln (3/2)z (D) none

Sol.

37. If $\lim\limits_{x\to\infty}\left(1+\dfrac{a}{x}+\dfrac{b}{x^2}\right)^{2x} = e^2$, then the values of a & b are

(A) $a \in R$, $b \in R$ (B) a = 1, $b \in R$

(C) $a \in R$, b = 2 (D) a = 1,b = 2

Sol.

38. $\lim\limits_{x\to\infty}\dfrac{\log_x n-[x]}{[x]}$, $n \in N$

(where [$*$] denotes greatest integer function)

(A) has value – 1 (B) has value 0

(C) has value 1 (D) does not exists

Sol.

39. $\lim\limits_{x\to\infty}\left(\dfrac{x^2+5x+3}{x^2+x+3}\right)^x$ is equal to

(A) e^4 (B) e^2 (C) e^3 (D) e

Sol.

40. Let α and β be the distinct roots of $ax^2 + bx + c = 0$, then $\lim\limits_{x\to\alpha}\dfrac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}$ is equal to

(A) $\dfrac{1}{2}(\alpha-\beta)^2$ (B) $-\dfrac{a^2}{2}(\alpha-\beta)^2$

(C) 0 (D) $\dfrac{a^2}{2}(\alpha-\beta)^2$

Sol.

41. $\lim\limits_{x\to 0}\dfrac{2\left(\sqrt{3}\sin\left(\dfrac{\pi}{6}+x\right)-\cos\left(\dfrac{\pi}{6}+x\right)\right)}{x\sqrt{3}(\sqrt{3}\cos x-\sin x)}$

(A) −1/3 (B) 2/3 (C) 4/3 (D) −4/3

Sol.

42. $\lim\limits_{n\to\infty}\dfrac{1.n+2(n-1)+3(n-2)+....+n.1}{1^2+2^2+3^2+....+n^2}$ has the value

(A) $\dfrac{1}{2}$ (B) $\dfrac{1}{3}$ (C) $\dfrac{1}{4}$ (D) 1

Sol.

43. Let $(\tan\alpha)\,x + (\sin\alpha)\,y = \alpha$ and $(\alpha\,\text{cosec}\,\alpha)\,x + (\cos\alpha)\,y = 1$ be two variable straight lines, α being the parameter. Let P be the point of intersection of the lines. In the limiting position when $\alpha \to 0$, the coordinates of P are

(A) (2, 1) (B) (2, −1) (C) (−2, 1) (D) (−2, −1)

Sol.

44. $\lim\limits_{x\to 0}\left(\cot\left(\dfrac{\pi}{4}+x\right)\right)^{\text{cosec}\,x} =$

(A) e^{-1} (B) e^2 (C) e^{-2} (D) e^1

Sol.

45. $\lim\limits_{x\to 0}\dfrac{\sin[\cos x]}{1+[\cos x]}$

(where [*] denotes greatest integer function)

(A) equal to 1 (B) sin 1

(C) equal to zero (D) non existent

Sol.

46. $\lim_{x\to\infty}\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)^x$ is

(A) e (B) e^2 (C) 1/e (D) does not exist

Sol.

47. If $\lim_{x\to 0}\frac{e^{-nx}+e^{nx}-2\cos\frac{nx}{2}-kx^2}{(\sin x-\tan x)}$ exists and finite, then possible values of 'n' and 'k' is :

(A) k = 3, n = 3 (B) k = 3, n = –2
(C) k = 5, n = 2 (D) k = –5, n = 2

Sol.

48. If $A_j = \frac{x-a_j}{|x-a_j|}$, j = 1, 2, ...n and

$a_1 < a_2 < a_3 < .... < a_n$ $\lim_{x\to a_m}$ $(A_1 . A_2. ...A_n)$, $1 \le m \le n$

(A) is equal to $(-1)^{n-m+1}$ (B) is equal to $(-1)^{n-m}$
(C) is equal to $(-1)^m$ (D) does not exist

Sol.

49. Let f(x) = $\frac{\ell n(x^2+e^x)}{\ell n(x^4+e^{2x})}$. If $\lim_{x\to\infty}$ f(x) = ℓ and $\lim_{x\to-\infty}$ f(x) = m then

(A) ℓ = m (B) ℓ = 2m (C) 2ℓ = m (D) ℓ + m = 0

Sol.

50. Let a = min {x^2 + 2x + 3, x ∈ R) & b = $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}$.

The value of $\sum_{r=0}^{n} a^r b^{n-r}$ is

(A) $\frac{2^{n+1}-1}{3.2^n}$ (B) $\frac{2^{n+1}+1}{3.2^n}$

(C) $\frac{4^{n+1}-1}{3.2^n}$ (D) None of these

Sol.

51. $\lim_{x\to\infty}\left(\frac{\cosh\frac{\pi}{x}}{\cos\frac{\pi}{x}}\right)^{x^2}$ where cosh t = $\frac{e^t+e^{-t}}{2}$ is equal to

(A) $e^{\frac{\pi^2}{2}}$ (B) e^{π^2} (C) $e^{\frac{3\pi^2}{2}}$ (D) $e^{2\pi^2}$

Sol.

52. $\lim\limits_{x \to 0^+} \dfrac{\left(\dfrac{\pi}{2} - \cot^{-1}\{x\}\right)x}{\text{sgn}(x) - \cos x}$ is

(where [∗] denotes fractional part function)

(A) 2 (B) 1

(C) 0 (D) does not exist

Sol.

53. $\lim\limits_{n \to \infty}$ $[(1 + x)\ (1 + x^2)\ (1 + x^4)\ .....(1 + x^{2^n})]$

if |x| < 1 has the value equal to

(A) 0 (B) 1 (C) 1 – x (D) $(1 - x)^{-1}$

Sol.

54. If $\lim\limits_{x \to 0} \dfrac{x^3}{\sqrt{a + x}(bx - \sin x)} = 1$ then the constants 'a' and 'b' are (where a > 0)

(A) b = 1, a = 36 (B) a = 1, b= 6

(C) a = 1, b = 36 (D) b = 1, a = 6

Sol.

55. $\lim\limits_{n \to \infty} \dfrac{1^2 n + 2^2(n-1) + 3^2(n-2) + ... + n^2.1}{1^3 + 2^3 + 3^3 + ... + n^3}$

(A) 1/3 (B) 2/3 (C) 1/2 (D) 1/6

Sol.

56. $\lim\limits_{x \to 0} \dfrac{\sin(6x^2)}{\ell n \cos(2x^2 - x)} =$

(A) 12 (B) –12 (C) 6 (D) –6

Sol.

57. If f(x) = $\sum\limits_{\lambda=1}^{n} \left(x - \dfrac{1}{\lambda}\right)\left(x - \dfrac{1}{\lambda + 1}\right)$ then $\lim\limits_{n \to \infty}$ f(0) is

(A) 1 (B) –1

(C) 2 (D) None of these

Sol.

58. $\text{Limit}\limits_{x \to 0} \dfrac{e^{-x^2/2} - \cos x}{x^3 \sin x} =$

(A) 1/4 (B) 1/6 (C) 1/12 (D) 1/8

Sol.

EXERCISE – II   MULTIPLE CORRECT (OBJECTIVE QUESTIONS)

1. Let f(x) = $\frac{x^2 - 9x + 20}{x - [x]}$ then
(where [∗] denotes greatest integer function)

(A) $\underset{x \to 5^-}{\text{Limit}}$ f(x) = 0   (B) $\underset{x \to 5^+}{\text{Limit}}$ f(x) = 1

(C) $\underset{x \to 5}{\text{Limit}}$ f(x) does not exist   (D) none of these

Sol.

2. Let f(x) = $\frac{\cos 2 - \cos 2x}{x^2 - |x|}$, then

(A) $\underset{x \to -1}{\text{Lim}}$ f(x) = 2 sin 2   (B) $\underset{x \to 1}{\text{Lim}}$ f(x) = 2 sin 2

(C) $\underset{x \to -1}{\text{Lim}}$ f(x) = 2 cos 2   (D) $\underset{x \to 1}{\text{Lim}}$ f(x) 2 cos 2

Sol.

3. Let f(x) = $\frac{\sqrt{x^2 + 2}}{3x - 6}$ =

(A) $\lim\limits_{x \to -\infty}$ f(x) = $-\frac{1}{3}$ (B) $\lim\limits_{x \to \infty}$ f(x) = $\frac{1}{3}$

(C) $\lim\limits_{x \to -\infty}$ f(x) = $\frac{1}{3}$ (D) $\lim\limits_{x \to \infty}$ f(x) = $-\frac{1}{3}$

Sol.

4. If $\underset{x \to 0}{\text{Limit}}$ (cos x + a sin b x)$^{1/x}$ = e^2, then the possible value of 'a' & 'b' is

(A) a = 1, b = 2   (B) a = 2, b = 1
(C) a = 3, b = 2/3   (D) a = 2/3, b = 3

Sol.

5. If l = $\underset{x \to 0}{\text{Lim}} \frac{x(1 + a\cos x) - b\sin x}{x^3}$ = $\underset{x \to 0}{\text{Lim}} \frac{1 + a\cos x}{x^2}$ – $\lim\limits_{x \to 0} \frac{b\sin x}{x^3}$, where $l \in$ R, then

(A) (a, b) = (–1, 0)   (B) a & b are nay real numbers

(C) l = 0   (D) l = $\frac{1}{2}$

Sol.

6. If $\underset{x \to 0}{\text{Limit}} \frac{\sin 2x + a\sin x}{x^3}$ = p (finite), then

(A) a = –2 (B) a = –1   (C) p = –2   (D) p = –1

Sol.

7. If $\underset{x \to 0}{\text{Limit}}$ (1 + ax + bx^2)$^{2/x}$ = e^3, then find values of a and b

(A) a = 3, b = 0   (B) a = 3/2, b = 1/2
(C) a = 3/2, b = 3/2   (D) a = 3/2, b = 0

Sol.

8. $\lim\limits_{x\to\infty}\dfrac{(ax+1)^n}{x^n+A} =$

(A) a^n if $n \in N$

(B) ∞ if $n \in Z$ & $a = A = 0$

(C) $\dfrac{1}{1+A}$ if $n = 0$

(D) a^n if $n \in Z$, $A = 0$ & $a \neq 0$

Sol.

9. The value of $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x^n}\right)^x$ is

(A) 1, $\forall$ n > 1

(B) e, $\forall$ n > 0

(C) ∞, $\forall n \in (0, 1)$

(D) 0, $\forall$ n > 1

Sol.

10. Let α, β be the roots of $ax^2 + bx + c = 0$, where $1 < \alpha < \beta$. Then $\lim\limits_{x\to x_0}\dfrac{|ax^2+bx+c|}{ax^2+bx+c} = 1$ then which of the following statements is incorrect

(A) a > 0 and $x_0 < 1$

(B) a > 0 and $x_0 < \beta$

(C) a < 0 and $\alpha < x_0 < \beta$

(D) a < 0 and $x_0 < 1$

Sol.

EXERCISE – III SUBJECTIVE QUESTIONS

1. $\lim\limits_{x\to 1}\dfrac{\sqrt[13]{x}-\sqrt[7]{x}}{\sqrt[5]{x}-\sqrt[3]{x}}$

Sol.

2. $\lim\limits_{x\to 1}\dfrac{x^2 - x.\ell n x + \ell n x - 1}{x-1}$

Sol.

3. $\lim\limits_{x\to 1}\dfrac{\left[\sum\limits_{k=1}^{100}x^k\right]-100}{x-1}$

Sol.

4. $\lim\limits_{x\to \frac{\pi}{4}}\dfrac{1-\tan x}{1-\sqrt{2}\sin x}$

Sol.

5. $\lim_{x\to 1}\left(\frac{p}{1-x^p}-\frac{q}{1-x^q}\right)$; p, q $\in$ N

Sol.

6. Find the sum of an infinite geometric series whose first term is the limit of the function f(x) = $\frac{\tan x-\sin x}{\sin^3 x}$ as $x \to 0$ and whose common ratio is the limit of the function g(x) = $\frac{1-\sqrt{x}}{(\cos^{-1}x)^2}$ as $x \to 1$.

Sol.

7. $\lim_{x\to\infty}$ (x – ℓn cosh x) where cosh t = $\frac{e^t+e^{-t}}{2}$.

Sol.

8. (a) $\lim_{x\to\frac{1}{\sqrt{2}}}\frac{\cos^{-1}\left(2x\sqrt{1-x^2}\right)}{x-\frac{1}{\sqrt{2}}}$

Sol.

(b) $\lim_{x\to\frac{\pi}{4}}\frac{\sqrt{1-\sqrt{\sin 2x}}}{\pi-4x}$

Sol.

(c) $\lim_{x\to -7}\frac{[x]^2+15[x]+56}{\sin(x+7)\sin(x+8)}$

(where [$*$] denotes greatest integer function)

Sol.

9. $\lim_{x\to\frac{3\pi}{4}}\frac{1+\sqrt[3]{\tan x}}{1-2\cos^2 x}$

Sol.

10. $\lim_{x\to 0}\frac{8}{x^8}\left[1-\cos\frac{x^2}{2}-\cos\frac{x^2}{4}+\cos\frac{x^2}{2}\cos\frac{x^2}{4}\right]$

Sol.

11. $\lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}-\cos\theta-\sin\theta}{(4\theta-\pi)^2}$

Sol.

12. $\lim_{h\to 0}\frac{\sin\left(\frac{\pi}{3}+4h\right)-4\sin\left(\frac{\pi}{3}+3h\right)+6\sin\left(\frac{\pi}{3}+2h\right)-4\sin\left(\frac{\pi}{3}+h\right)+\sin\frac{\pi}{3}}{h^4}$

Sol.

13. $\lim_{x \to \infty} x^2\left(\sqrt{\frac{x+2}{x}} - \sqrt[3]{\frac{x+3}{x}}\right)$

Sol.

14. $\lim_{x \to 0}$ [ℓn (1 + sin²x).cot(ℓn² (1 + x))]

Sol.

15. Find a & b if

(i) $\lim_{x \to \infty}\left[\frac{x^2+1}{x+1} - ax - b\right] = 0$

Sol.

(ii) $\lim_{x \to -\infty}\left[\sqrt{x^2-x+1} - ax - b\right] = 0$

Sol.

16. $\lim_{x \to -\infty} \frac{(3x^4+2x^2)\sin\frac{1}{x} + |x|^3 + 5}{|x|^3 + |x|^2 + |x| + 1}$

Sol.

17. If $\ell = \lim_{n \to \infty} \sum_{r=2}^{n}\left((r+1)\sin\frac{\pi}{r+1} - r\sin\frac{\pi}{r}\right)$ then find $\{\ell\}$. (where { } denotes the fractional part function)

Sol.

18. $\lim_{x \to 1} \frac{(\ell n(1+x) - \ell n2)(3.4^{x-1} - 3x)}{[(7+x)^{\frac{1}{3}} - (1+3x)^{\frac{1}{2}}].\sin(x-1)}$

Sol.

19. $\lim_{x \to 0} \frac{27^x - 9^x - 3^x + 1}{\sqrt{2} - \sqrt{1+\cos x}}$

Sol.

20. Let f(x) = $\frac{x}{\sin x}$, x > 0 and g(x) = x + 3, x < 1
= 2 – x, x ≤ 0 = x² – 2x – 2, 1 ≤ x < 2
= x – 5, x ≥ 2

find LHL and RHL of g(f(x)) at x = 0 and hence find $\lim_{x \to 0}$ g(f(x)).

Sol.

21. (a) $\lim_{x \to 0} \tan^{-1}\frac{a}{x^2}$, where (a > 0 & a ∈ R)

Sol.

(b) Plot the graph of the function

f(x) = $\lim_{t \to 0}\left(\frac{2x}{\pi}\tan^{-1}\frac{x}{t^2}\right)$

Sol.

22. Let $P_n = a^{P_{n-1}} - 1$, $\forall$ n = 2, 3, ... and Let $P_1 = a^x - 1$ where $a \in R^+$ then evaluate $\lim_{x\to 0} \frac{P_n}{x}$.

Sol.

23. If the $\lim_{x\to 0} \frac{1}{x^3}\left(\frac{1}{\sqrt{1+x}} - \frac{1+ax}{1+bx}\right)$ exists and has the value equal to ℓ, then find the value of $\frac{1}{a} - \frac{2}{\ell} + \frac{3}{b}$.

Sol.

24. $\lim_{y\to 0}\left[\underset{x\to\infty}{\text{Limit}} \frac{\exp\left(x\,\ell n(1+\frac{ay}{x})\right) - \exp\left(x\,\ell n(1+\frac{by}{x})\right)}{y}\right]$

Sol.

25. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences such that
(i) $a_n + b_n + c_n = 2n + 1$; **(ii)** $a_nb_n + b_nc_n + c_na_n = 2n - 1$
(iii) $a_nb_nc_n = -1$; **(iv)** $a_n < b_n < c_n$
Then find the value of $\lim_{n\to\infty} (na_n)$.

Sol.

26. If $n \in N$ and $a_n = 2^2 + 4^2 + 6^2 + \ldots\ldots + (2n)^2$ and $b_n = 1^2 + 3^2 + 5^2 + \ldots\ldots + (2n-1)^2$. Find the value $\lim_{n\to\infty} \frac{\sqrt{a_n} - \sqrt{b_n}}{\sqrt{n}}$

Sol.

27. $\lim_{x\to\infty}\left[\frac{2x^2+3}{2x^2+5}\right]^{8x^2+3}$

Sol.

28. $\lim_{x\to\infty}\left(\frac{x+c}{x-c}\right)^x = 4$ then find c

Sol.

29. $\lim_{x\to 1}\left(\tan\frac{\pi x}{4}\right)^{\tan\frac{\pi x}{2}}$

Sol.

30. $\lim_{x\to 0}\left(\frac{x-1+\cos x}{x}\right)^{\frac{1}{x}}$

Sol.

EXERCISE – IV ADVANCED SUBJECTIVE QUESTIONS

1. $\lim\limits_{x\to\infty} x^2 \sin\left(\ell n\sqrt{\cos\frac{\pi}{x}}\right)$

Sol.

2. $\lim\limits_{x\to\infty}\left[\cos\left(2\pi\left(\frac{x}{1+x}\right)^a\right)\right]^{x^2}$ $a \in Q$

Sol.

3. Let $f(x) = \frac{\sin^{-1}(1-\{x\}).\cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}.(1-\{x\})}$ then find $\lim\limits_{x\to 0^+}$ f(x) and $\lim\limits_{x\to 0^-}$ f(x), where {x} denotes the fractional part function.

Sol.

4. $\lim\limits_{n\to\infty}\left(\frac{\sqrt{n^2+n}-1}{n}\right)^{2\sqrt{n^2+n}-1}$

Sol.

5. $\lim\limits_{x\to\infty}\left(\frac{a_1^{\frac{1}{x}}+a_2^{\frac{1}{x}}+a_3^{\frac{1}{x}}+.....+a_n^{\frac{1}{x}}}{n}\right)^{nx}$

where $a_1, a_2, a_3, ... a_n > 0$

Sol.

6. $\lim\limits_{x\to 0}\left[\frac{(1+x)^{1/x}}{e}\right]^{1/x}$

Sol.

7. $\lim\limits_{x\to\infty}\left(\frac{\cosh(\pi/x)}{\cos(\pi/x)}\right)^{x^2}$ where $\cosh t = \frac{e^t+e^{-t}}{2}$

Sol.

8. $\lim\limits_{x\to a}\frac{1}{(a^2-x^2)^2}\left(\frac{a^2+x^2}{ax}-2\sin\left(\frac{a\pi}{2}\right)\sin\left(\frac{\pi x}{2}\right)\right)$ where a is an odd integer

Sol.

9. If $L = \lim\limits_{x\to 1}\frac{(1-x)(1-x^2)(1-x^3)......(1-x^{2n})}{[(1-x)(1-x^2)(1-x^3).......(1-x^n)]^2}$ then show that L can be equal to

(a) $\prod_{r=1}^{n} \frac{n+r}{r}$

Sol.

(b) $\frac{1}{n!}\prod_{r=1}^{n}(4r-2)$

Sol.

(c) The sum of the coefficients of two middle terms in the expansion of $(1 + x)^{2n-1}$.

Sol.

(d) The coefficient of x^n in the expansion of $(1 + x)^{2n}$.

Sol.

10. If $\lim_{x\to\infty} \frac{a(2x^3-x^2)+b(x^3+5x^2-1)-c(3x^3+x^2)}{a(5x^4-x)-bx^4+c(4x^4+1)+2x^2+5x} = 1$, then the value of (a + b + c) can be expressed in the lowest form as $\frac{p}{q}$. Find the value of (p + q).

Sol.

11. Let $x_0 = 2\cos\frac{\pi}{6}$ and

$x_n = \sqrt{2+x_{n-1}}$, n = 1, 2, 3, ..., find $\lim_{n\to\infty} 2^{(n+1)}.\sqrt{2-x_n}$.

Sol.

12. $\lim_{x\to 0}\left[\frac{\ell n(1+x)^{1+x}}{x^2}-\frac{1}{x}\right]$

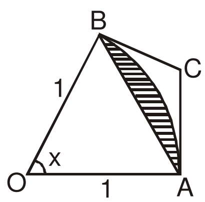
Sol.

13. Let L = $\prod_{n=3}^{\infty}\left(1-\frac{4}{n^2}\right)$; M = $\prod_{n=2}^{\infty}\left(\frac{n^3-1}{n^3+1}\right)$ and

$N=\prod_{n=1}^{\infty}\frac{(1+n^{-1})^2}{1+2n^{-1}}$, then find the value of $L^{-1} + M^{-1} + N^{-1}$.

Sol.

14. A circular arc of radius 1 subtends an angle of x radians, $0 < x < \frac{\pi}{2}$ as shown in the figure. The point C is the intersection of the two tangent lines at A & B. Let T(x) be the area of triangle ABC & let S(x) be the area of the shaded region. Compute :



(a) T(x)

Sol.

(b) S(x)

Sol.

(c) the limit of $\frac{T(x)}{S(x)}$ as $x \to 0$.

Sol.

15. Let $f(x) = \lim_{n\to\infty} \sum_{n=1}^{n} 3^{n-1}\sin^3\frac{x}{3^n}$ and

$g(x) = x - 4f(x)$. Evaluate $\lim_{x\to 0} (1 + g(x))^{\cot x}$.

Sol.

16. If $f(n,\theta)=\prod_{r=1}^{n}\left(1-\tan^2\frac{\theta}{2^r}\right)$, then compute $\lim_{n\to\infty} f(n, \theta)$

Sol.

17. $L = \lim_{x\to 0} \frac{\sqrt{\frac{\cos 2x+(1+3x)^{1/3}}{2}} - \sqrt[3]{\frac{4\cos^3 x-\ell n(1+x)^4}{4}}}{x}$.

If L=a/b where 'a' and 'b' are relatively primes find (a+b).

Sol.

18. f(x) is the function such that $\lim_{x\to 0}\frac{f(x)}{x} = 1$. If $\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{(f(x))^3} = 1$, then find the value of a and b.

Sol.

19. Through a point A on a circle, a chord AP is drawn & on the tangent at A a point T is taken such that AT = AP. If TP produced meet the diameter through A at Q, prove that the limiting value of AQ when P moves upto A is double the diameter of the circle.

Sol.

20. At the end points A, B of the fixed segment of length L, lines are drawn meeting in C and making angles θ and 2θ respectively with the given segment. Let D be the foot of the altitude CD and let x represents the length of AD. Find the value of x as θ tends to zero i.e. $\lim_{\theta\to 0} x$.

Sol.

21. At the end-points and the midpoint of a circular arc AB tangent lines are drawn, and the points A and B are joined with a chord. Prove that the ratio of the areas of the two triangles thus formed tends to 4 as the arc AB decreases indefinitely.

Sol.

22. If L = $\lim_{x\to 0}\left(\frac{1}{\ell n(1+x)}-\frac{1}{\ell n(x+\sqrt{1+x^2})}\right)$ then find the value of $\frac{L+153}{L}$.

Sol.

23. Let f(x) = $\lim_{n\to\infty}\frac{2x^{2n}\sin\frac{1}{x}+x}{1+x^{2n}}$ then find

(a) $\lim_{x\to\infty} x\,f(x)$

Sol.

(b) $\lim_{x\to 1} f(x)$

Sol.

(c) $\lim_{x\to 0} f(x)$

Sol.

(d) $\lim_{x\to -\infty} f(x)$

Sol.

24. Using Sandwich theorem, evaluate

(a) $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+........+\frac{1}{\sqrt{n^2+2n}}\right)$

Sol.

(b) $\lim_{n\to\infty}\frac{1}{1+n^2}+\frac{2}{2+n^2}+...........+\frac{n}{n+n^2}$

Sol.

25. The sequence $\{a_n\}_{n=1}^{+\infty}$ is defined by $a_1 = 0$ and $a_{n+1} = a_n + 4n + 3$, $n \ge 1$.

Find the value of $\operatorname{Limit}_{n\to+\infty}\frac{\sqrt{a_n}+\sqrt{a_{4n}}+\sqrt{a_{4^2n}}+\sqrt{a_{4^3n}}+....+\sqrt{a_{4^{10}n}}}{\sqrt{a_n}+\sqrt{a_{2n}}+\sqrt{a_{2^2n}}+\sqrt{a_{2^3n}}+....+\sqrt{a_{2^{10}n}}}$.

Sol.

EXERCISE – V JEE PROBLEMS

1. For $x \in R$, $\lim_{x\to\infty}\left(\frac{x-3}{x+2}\right)^x =$ **[JEE 2000 (Scr.)]**

(A) e (B) e^{-1} (C) e^{-5} (D) e^5

Sol.

2. $\lim_{x\to 0}\frac{\sin(\pi\cos^2 x)}{x^2}$ equals **[JEE 2001 (Scr.)]**

(A) $-\pi$ (B) π (C) $\frac{\pi}{2}$ (D) 1

Sol.

3. Evaluate $\lim_{x\to 0}\frac{a^{\tan x}-a^{\sin x}}{\tan x-\sin x}$, $a > 0$. **[REE 2001, 3]**

Sol.

4. The integer n for which $\lim_{x\to 0}\frac{(\cos x-1)(\cos x-e^x)}{x^n}$ is a finite non-zero number is **[JEE 2002 (Scr.), 3]**

(A) 1 (B) 2 (C) 3 (D) 4

Sol.

5. If $\lim_{x\to 0}\frac{\sin(nx)[(a-n)nx-\tan x]}{x^2} = 0$ $(n > 0)$ then the value of 'a' is equal to **[JEE 2003 (Scr.)]**

(A) $\frac{1}{n}$ (B) $n^2 + 1$ (C) $\frac{n^2+1}{n}$ (D) None

Sol.

6. Find the value of $\lim_{n\to\infty}\left[\frac{2}{\pi}(n+1)\cos^{-1}\left(\frac{1}{n}\right)-n\right]$.

[JEE 2004, 2]

Sol.

7. Let $L = \lim_{x\to 0}\frac{a-\sqrt{a^2-x^2}-\frac{x^2}{4}}{x^4}$, $a > 0$, If L is finite, then **[JEE 2009, 4]**

(A) $a = 2$ (B) $a = 1$ (C) $L = \frac{1}{64}$ (D) $L = \frac{1}{32}$

Sol.

8. If $\lim_{x\to 0}[1+x\ln(1+b^2)]^{\frac{1}{x}} = 2b\sin^2\theta$, $b > 0$ and $\theta \in (-\pi, \pi]$. Then the value of θ is **[JEE 2011, 3]**

(A) $\pm\frac{\pi}{4}$ (B) $\pm\frac{\pi}{3}$ (C) $\pm\frac{\pi}{6}$ (D) $\pm\frac{\pi}{2}$

Sol.

Answer Ex–I — SINGLE CORRECT (OBJECTIVE QUESTIONS)

1. C	**2.** D	**3.** D	**4.** B	**5.** D	**6.** C	**7.** C	**8.** C	**9.** A
10. C	**11.** B	**12.** C	**13.** A	**14.** A	**15.** B	**16.** C	**17.** C	**18.** D
19. D	**20.** B	**21.** C	**22.** C	**23.** B	**24.** C	**25.** A	**26.** B	**27.** A
28. B	**29.** C	**30.** C	**31.** A	**32.** D	**33.** C	**34.** A	**35.** B	**36.** B
37. B	**38.** A	**39.** A	**40.** D	**41.** C	**42.** A	**43.** B	**44.** C	**45.** C
46. A	**47.** C	**48.** D	**49.** A	**50.** C	**51.** B	**52.** A	**53.** D	**54.** A
55. A	**56.** B	**57.** A	**58.** C					

Answer Ex–II — MULTIPLE CORRECT (OBJECTIVE QUESTIONS)

1. A,B,C **2.** A,B **3.** A,B **4.** A,B,C,D **5.** A,D **6.** A,D **7.** B,C,D **8.** A,B,C,D **9.** A,C
10. B,D

Answer Ex–III — SUBJECTIVE QUESTIONS

1. $\frac{45}{91}$ **2.** 2 **3.** 5050 **4.** 2 **5.** $\frac{p-q}{2}$

6. $a = \frac{1}{2}$; $r = \frac{1}{4}$; $S = \frac{2}{3}$ **7.** $\ell n\, 2$ **8.** **(a)** does not exist ; **(b)** does not exist ; **(c)** 0

9. $-\frac{1}{3}$ **10.** $\frac{1}{32}$ **11.** $\frac{1}{16\sqrt{2}}$ **12.** $\frac{\sqrt{3}}{2}$ **13.** 1/2 **14.** 1

15. **(i)** $a = 1, b = -1$ **(ii)** $a = -1, b = \frac{1}{2}$ **16.** -2 **17.** $\pi - 3$ **18.** $-\frac{9}{4}\ \ell n \frac{4}{e}$

19. $8\sqrt{2}(\ell n\, 3)^2$ **20.** $-3, -3, -3$ **21.** **(a)** $\pi/2$ if $a > 0$; 0 if $a = 0$ and $-\pi/2$ if $a < 0$; **(b)** $f(x) = |x|$

22. $(\ell n\, a)^n$ **23.** 72 **24.** $a - b$ **25.** $-1/2$ **26.** $\frac{\sqrt{3}}{2}$ **27.** e^{-8}

28. $c = \ell n\, 2$ **29.** e^{-1} **30.** $e^{-1/2}$

Answer Ex–IV | ADVANCED SUBJECTIVE QUESTIONS

1. $-\frac{\pi^2}{4}$ **2.** $e^{-2\pi^2a^2}$ **3.** $\frac{\pi}{2}, \frac{\pi}{2\sqrt{2}}$ **4.** e^{-1} **5.** $(a_1.a_2.a_3 ......a_n)$ **6.** $e^{-\frac{1}{2}}$

7. e^{π^2} **8.** $\frac{\pi^2a^2+4}{16a^4}$ **10.** 167 **11.** $\frac{\pi}{3}$ **12.** 1/2 **13.** 8

14. $T(x) = \frac{1}{2}\tan^2\frac{x}{2}.\sin x$ or $\tan\frac{x}{2} - \frac{\sin x}{2}$, $S(x) = \frac{1}{2}x - \frac{1}{2}\sin x$, limit = $\frac{3}{2}$

15. $g(x) = \sin x$ and $\ell = e$ **16.** $\frac{\theta}{\tan\theta}$ **17.** 19 **18.** a = – 5/2, b = –3/2

20. $\frac{2L}{3}$ **21.** 4 **22.** 307 **23.** (a) 2, (b) D.N.E., (c) 0, (d) 0

24. (a) 2 ; (b) 1/2 **25.** 683

Answer Ex–V | JEE PROBLEMS

1. C **2.** B **3.** $\ell n\ a$ **4.** C **5.** C **6.** $1 - \frac{2}{\pi}$ **7.** A, C

8. D